SOME VARIETIES CONTAINING RELATION ALGEBRAS

BY

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ABSTRACT. Three varieties of algebras are introduced which extend the variety RA of relation algebras. They are obtained from RA by weakening the associative law for relative product, and are consequently called nonassociative, weakly-associative and semiassociative relation algebras, or NA, WA, and SA, respectively. Each of these varieties arises naturally in solving various problems concerning relation algebras. We show, for example, that WA is the only one of these varieties which is closed under the formation of complex algebras of atom structures of algebras, and that WA is the closure of the variety of representable RA's under relativization. The paper also contains a study of the elementary theories of these varieties, various representation theorems, and numerous examples.

0. Introduction. Relation algebras (RA's) have a binary operation ; which serves as an abstract algebraic analogue of the relative product of binary relations. (The relative product of R, $S \subseteq U \times U$, is $R \mid S = \{\langle x, z\rangle : \langle x, y\rangle \in R$ and $\langle y, z\rangle \in S$ for some $y \in U\}$.) The relative product is associative, and one of the postulates for RA's is that ; is associative. The associativity of relative product can be expressed by a sentence in a first-order language with binary relation symbols, namely

$$(1) \qquad \forall_x \forall_y \left[\exists_z \left(\exists_y (Rxy \wedge Syz) \wedge Tzy \right) \leftrightarrow \exists_z \left(Rxz \wedge \exists_x (Szx \wedge Txy) \right) \right].$$

Although this sentence has three variables, it cannot be proved from the ordinary axioms of first-order logic without using four variables. In contrast, all the other postulates for relation algebras can not only be expressed but proved using only three variables. (These facts were first proved by Tarski. For a proof that (1) requires four variables to prove, see [3].) Tarski asked whether there are any other equations whose translations into first-order sentences can be expressed and proved using only three variables, but which are not derivable from the postulates for RA's without using the associativity of ;. There are such equations. One of them is a special case of the associative law for ; called the "semiassociative law",

$$(2) \qquad\qquad x; (1;1) = (x;1);1.$$

The class SA of semiassociative relation algebras is defined by the postulates for RA's with the associative law for ; replaced by (2). This class properly includes RA, as will be shown in this paper.

About 90% of the theorems for RA's proved in Part I of [1] for RA's carry over to SA's. In fact, about 80% of those theorems do not even require the semiassociative law, and hence apply to the even wider class of nonassociative relation algebras NA. The definition of NA is obtained from that of RA by omitting the associative law. NA's are Boolean algebras with operators in the sense of [6, Definition 2.13]. Consequently the Representation Theorem 3.10 of [6] can be applied. One of the goals of this paper is to obtain a more explicit form of this theorem for NA, SA, and RA. To do so we introduce the appropriate notions of complex algebra and atom structure, and study some of their basic properties. In the course of this study another class of algebras naturally arises, namely WA, the weakly-associative relation algebras. The definition of WA is obtained from that of RA by replacing the associative law for ; by

$$(3) \qquad\qquad (1'\cdot x);(1;1) = ((1'\cdot x);1);1.$$

Note that (3) is a special case of (2). We shall see that $RA \subset SA \subset WA \subset NA$. Another goal of this paper is to introduce the notion of relativization for NA's and prove a new result for RA's involving relativization. It turns out, however, that WA is exactly the class of algebras for which this theorem holds.

The paper is organized as follows. In §1 we define the algebras which concern us and prove a few elementary properties of NA's. The duality theorems for RA's in [1] carry over to NA's and are given in §1. In §2 we define complex algebras of certain relational structures and characterize those structures which give use to NA's, WA's, SA's, and RA's. Some examples appear at the end of the section. Atom structures are defined and studied in §3. Theorems 3.8–3.10 in that section are reasons for introducing WA. Examples of NA's which are not WA's also appear there. Results from §§2 and 3 are combined with some theorems of [6] in §4 to yield an explicit form of the Representation Theorem (4.3). §5 deals with relativization. The major result there is that WA is the class of subalgebras of relativized representable RA's (5.20).

In set-theoretical, logical, and general algebraic notation and terminology we follow [4]. For the theory of Boolean algebras our reference is [16]. The most important reference for this paper is [6]. We shall use several of the theorems in that paper, and refer the reader there for the definitions of Boolean algebra with operators, and of conjugate, self-conjugate, normal, monotone, additive, and completely additive operators on a Boolean algebra.

1. Definitions and elementary properties.

DEFINITION 1.1. $\mathfrak{A}$ is a *relation-type algebra* iff $\mathfrak{A} = \langle A, +, \cdot, -, 0, 1, ;, {}^{\cup}, 1'\rangle$, where $+$, $\cdot$, and ; are binary operations on A, $-$ and ${}^{\cup}$ are unary operations on A, and $0, 1, 1' \in A$. The reduct $\mathfrak{Bl}\,\mathfrak{A} = \langle A, +, \cdot, -, 0, 1\rangle$ is called the *Boolean part* of $\mathfrak{A}$. The Boolean operations of $\mathfrak{A}$ are $+$, $\cdot$, and $-$. For all $x, y \in A$ let $x \mathbin{\dot{+}} y = -((-x);(-y))$, and $0' = -1'$. $x \mathbin{\dot{+}} y$ is the *relative sum* of x and y, $x;y$ is the *relative product* of x and y, and $x^{\cup}$ is the *converse* of x. The *Peircean* (or *relative*) *operations* of $\mathfrak{A}$ are $+$, ;, and ${}^{\cup}$. The elements $0, 1, 0'$, and $1'$ are called the *zero element, unit element, diversity element*, and *identity element*, respectively. For each $x \in A$ let l_x,

r_x, $l_x^\cup$, and $r_x^\cup$ be the functions mapping A into A which are defined as follows for all $y \in A$:

$$l_x(y) = x;y, \qquad r_x(y) = y;x,$$
$$l_x^\cup(y) = x^\cup;y, \qquad r_x^\cup(y) = y;x^\cup.$$

Parentheses will be omitted from expressions involving the operations of a relation-type algebra according to the convention that the operations should be performed in the following order: $\cup$, $-$, $;$, $\cdot$, $\dagger$, $+$. Thus, for example, $-x^\cup = -(x^\cup)$, $-x;-y = (-x);(-y)$, and $v \dagger w; x \cdot y + z = (v \dagger ((w;x) \cdot y)) + z$. When the same binary operation occurs several times, the calculation proceeds from left to right, e.g. $x;y;z = (x;y);z$.

Let $\mathfrak{A}$ be a relation-type algebra. If $\mathfrak{Bl}\,\mathfrak{A}$ is a Boolean algebra, then we will apply terminology from the theory of Boolean algebras to $\mathfrak{A}$, with the understanding that we are referring to $\mathfrak{Bl}\,\mathfrak{A}$. For example, $\mathfrak{A}$ is complete iff $\mathfrak{Bl}\,\mathfrak{A}$ is complete, and $x \in A$ is an atom of $\mathfrak{A}$ iff x is an atom of $\mathfrak{Bl}\,\mathfrak{A}$.

DEFINITION 1.2. A *nonassociative relation algebra* is a relation-type algebra $\mathfrak{A} = \langle A, +, \cdot, -, 0, 1, ;, \cup, 1' \rangle$ which satisfies the following postulates:

(1) $\mathfrak{Bl}\,\mathfrak{A}$ is a Boolean algebra,

(2) $x = x;1' = 1';x$ for every $x \in A$,

(3) l_x and $l_x^\cup$ are conjugate for every $x \in A$, i.e. $x;y \cdot z = 0$ iff $x^\cup;z \cdot y = 0$ for all $x, y, z \in A$,

(4) r_x and $r_x^\cup$ are conjugate for every $x \in A$, i.e. $y;x \cdot z = 0$ iff $z;x^\cup \cdot y = 0$ for all $x, y, z \in A$.

NA is the class of all nonassociative relation algebras.

$\mathfrak{A}$ is a *weakly-associative relation algebra* iff $\mathfrak{A} \in NA$ and

(5) $\qquad\qquad (1' \cdot x);1;1 = (1' \cdot x);1$ for every $x \in A$.

WA is the class of all weakly-associative relation algebras.

$\mathfrak{A}$ is a *semiassociative relation algebra* iff $\mathfrak{A} \in NA$ and

(6) $\qquad\qquad\qquad x;1;1 = x;1$ for every $x \in A$.

SA is the class of all semiassociative relation algebras.

$\mathfrak{A}$ is a *relation algebra* iff $\mathfrak{A} \in NA$ and

(7) $\qquad\qquad\qquad x;y;z = (x;y);z$ for all $x, y, z \in A$.

RA is the class of all relation algebras.

COROLLARY 1.3. (1) $SA \subseteq WA \subseteq NA$.

(2) $RA \subseteq NA$.

We will use "*BA*", "*NA*", "*WA*", "*SA*", and "*RA*" as abbreviations of the phrases "Boolean algebra", "nonassociative relation algebra", "weakly-associative relation algebra", "semiassociative relation algebra," and "relation algebra", respectively. Postulates 1.2(2)–(7) will be referred to respectively as the *identity law* (IL), *left Peircean law* (LPL), *right Peircean law* (RPL), *weak associative law* (WL), *semiassociative law* (SL), and *associative law* (AL).

It will be shown below that $1 = 1;1$ in every NA. Hence, relative to postulates 1.2(1)–(4), the WL and SL are special cases of the AL, namely $(1'\cdot x);1;1 = (1'\cdot x);(1;1)$ and $x;1;1 = x;(1;1)$, respectively. This is the reason for the use of the terms "weakly-associative" and "semiassociative" in 1.2.

Postulates 2.1(1)–(4), (7) were first used as a definition of RA in [7]. The definition of RA in [1] is proved in Theorem 2.2 of [1] to be equivalent to the one adopted here.

THEOREM 1.4. *Let $\mathfrak{A}$ be a relation-type algebra with $\mathfrak{Bl}\,\mathfrak{A} \in BA$.*

(1) *The following conditions are equivalent:*

  (a) *$\mathfrak{A}$ satisfies the LPL,*

  (b) $x;(y \cdot -(x^{\cup};z)) \leqslant x;y \cdot -z$ *and* $x^{\cup};(y \cdot -(x;z)) \leqslant x^{\cup};y \cdot -z$ *for all x, y, $z \in A$,*

  (c) $x;0 = 0$, $x;y \cdot z \leqslant x;(y \cdot x^{\cup};z)$ *and* $x^{\cup};y \cdot z \leqslant x^{\cup};(y \cdot x;z)$ *for all x, y, $z \in A$.*

(2) *The following conditions are equivalent:*

  (a) *$\mathfrak{A}$ satisfies the RPL,*

  (b) $(y \cdot -(z;x^{\cup}));x \leqslant y;x \cdot -z$ *and* $(y \cdot -(z;x));x^{\cup} \leqslant y;x^{\cup} \cdot -z$ *for all x, y, $z \in A$,*

  (c) $0;x = 0$, $y;x \cdot z \leqslant (y \cdot z;x^{\cup});x$, *and* $y;x^{\cup} \cdot z \leqslant (y \cdot z;x);x^{\cup}$ *for all x, y, $z \in A$.*

PROOF. Use Theorem 1.15 of [6]. Notice that if l_x (or r_x) is normal for every $x \in A$, then so is $l_x^{\cup}$ (or $r_x^{\cup}$).

COROLLARY 1.5. *NA, WA, SA, and RA are finitely based equational classes; $HSPNA = NA$, $HSPWA = WA$, $HSPSA = SA$, and $HSPRA = RA$.*

PROOF. Use 1.4 and the fact that BA's can be characterized by equations when $+$, $\cdot$, $-$, 0, and 1 are taken as fundamental.

We now turn to examples of RA's which motivated the abstract definition of RA and gave rise to the concept of representability. Let U be a set. Notice that $\mathrm{Sb}(U \times U)$, the set of subsets of $U \times U$, is closed under $\cup$, $\cap$, $_{U \times U}\!\sim$ (complementation with respect to $U \times U$), $|$ (defined by $R \,|\, S = \{\langle u, w \rangle : \langle u, v \rangle \in R$ and $\langle v, w \rangle \in S$, for some $v \in U\}$ for all R, $S \subseteq U \times U$), and $^{-1}$ (where $R^{-1} = \{\langle v, u \rangle : \langle u, v \rangle \in R\}$). Furthermore, if E is an arbitrary binary relation, then $\mathrm{Sb}\,E$ is closed under $\cup$, $\cap$, $_E\!\sim$, $|$, and $^{-1}$ iff E is an equivalence relation.

DEFINITION 1.6. Let U be any set. Then $\mathrm{Re}\,U$ is the set of all (binary) relations on U, i.e., $\mathrm{Re}\,U = \mathrm{Sb}(U \times U)$, and

$$\mathfrak{Re}\,U = \left\langle \mathrm{Re}\,U, \cup, \cap, _{U \times U}\!\sim, \varnothing, U \times U, |, {}^{-1}, \mathrm{Id}_U \right\rangle$$

is the *algebra of relations on U.*

Let E be any equivalence relation. Then

$$\mathfrak{Sb}\,E = \left\langle \mathrm{Sb}\,E, \cup, \cap, _E\!\sim, \varnothing, E, |, {}^{-1}, \mathrm{Id}_{\mathrm{Fd}\,E} \right\rangle$$

is the *algebra of subrelations of E.*

THEOREM 1.7. *Let U be a set and E an equivalence relation.*

(1) $\Re e\, U \in RA$ *and* $\mathfrak{S}\mathfrak{b}\, E \in RA$.

(2) $\Re e\, U$ *is a one-element algebra iff* $U = \varnothing$, *and* $\mathfrak{S}\mathfrak{b}\, E$ *is a one-element algebra iff* $E = \varnothing$.

(3) *For every set* V, $\Re e\, V \cong \Re e\, U$ *iff* $|V| = |U|$.

(4) $\Re e\, U = \mathfrak{S}\mathfrak{b}\, E$ *iff* $U \times U = E$.

(5) $\mathfrak{S}\mathfrak{b}\, E \cong P_{U \in (\mathrm{Fd}\, E)/E}\, \Re e\, U$.

PROOF. Parts (1)–(4) are easy to check. For part (5), define a function f mapping Sb E into the direct product of the equivalence classes of E as follows: if $R \in$ Sb E then $f(R) = \langle R \cap (U \times U) : U \in (\mathrm{Fd}\, E)/E \rangle$. It is easy to check that f is an isomorphism.

DEFINITION 1.8. A RA $\mathfrak{A}$ is *representable* if it is isomorphic to a subalgebra of $\mathfrak{S}\mathfrak{b}\, E$, for some equivalence relation E. RRA is the class of all representable RA's, i.e. $RRA = IS\{\mathfrak{S}\mathfrak{b}\, E : E$ is an equivalence relation$\}$.

THEOREM 1.9. (1) $RRA \subseteq RA$.

(2) $\Re e\, U \in RRA$ *for every set* U.

(3) $\mathfrak{S}\mathfrak{b}\, E \in RRA$ *for every equivalence relation* E.

(4) $HSPRRA = RRA$.

PROOF. Parts (1)–(3) and $SPRRA = RRA$ are easy to prove using 1.7. For a proof of $HRRA = RRA$, see Theorem 2.5(i) of [17], or Corollary 11 of [11].

The rest of this section will be devoted to some elementary properties of NA's which will be needed in this paper.

THEOREM 1.10. *Let $\mathfrak{A}$ be a relation-type algebra in which the Peircean laws hold. Then* ; *is normal and completely additive.*

PROOF. The operation ; is normal and completely additive iff l_x and r_x are normal and completely additive for all $x \in A$, so the theorem follows immediately from Theorem 1.14 of [6].

THEOREM 1.11. *Let $\mathfrak{A} \in NA$. Then* $^\cup$ *is self-conjugate, normal, and completely additive.*

PROOF. In view of Theorem 1.14 of [6], it suffices to show $^\cup$ is self-conjugate, i.e. $x^\cup \cdot y = 0$ iff $x \cdot y^\cup = 0$ for all x, $y \in A$. Each of the following formulas is equivalent to the one preceding it (if any) for the reason given to the right.

$$0 = x^\cup \cdot y$$
$$0 = x^\cup ; 1' \cdot y \quad \text{IL}$$
$$0 = x ; y \cdot 1' \quad \text{LPL}$$
$$0 = 1' ; y^\cup \cdot x \quad \text{RPL}$$
$$0 = x \cdot y^\cup \quad \text{IL.}$$

THEOREM 1.12. *Every NA is a normal BA with operators.*

PROOF. Use 1.10 and 1.11.

THEOREM 1.13. *Let* $\mathfrak{A} \in NA$. *Then for all* $w, x, y, z \in A$ *we have*

(1) $(x + y); z = x; z + y; z,$

(2) $x;(y + z) = x; y + x; z,$

(3) *if* $w \le x$ *and* $y \le z$, *then* $w; y \le x; z$,

(4) $x \le x; 1$ *and* $x \le 1; x,$

(5) $1 = 1; 1,$

(6) $(x + y)^{\cup} = x^{\cup} + y^{\cup},$

(7) $0^{\cup} = 0,$

(8) $x^{\cup\cup} = x,$

(9) $1^{\cup} = 1,$

(10) $(-x)^{\cup} = -x^{\cup},$

(11) $(x \cdot y)^{\cup} = x^{\cup} \cdot y^{\cup},$

(12) $x \le y$ *iff* $x^{\cup} \le y^{\cup},$

(13) $(x; y)^{\cup} = y^{\cup}; x^{\cup},$

(14) $1'^{\cup} = 1'$ *and* $0'^{\cup} = 0',$

(15) *if* $x, y \le 1'$ *then* $x; y = x \cdot y,$

(16) *if* $x \le 1'$ *then* $x^{\cup} = x.$

PROOF. (1)–(3) follow immediately from 1.10. Using (3) and the identity law, we get $x = x; 1' \le x; 1$ and $x = 1'; x \le 1; x$, so (4) holds. Since $1; 1 \le 1$, we get (5) from (4). (6) and (7) follow from 1.11. For (8) we first notice that the following formulas are equivalent:

$$0 = x \cdot y$$
$$0 = x; 1' \cdot y \qquad \text{IL}$$
$$0 = x^{\cup}; y \cdot 1' \qquad \text{LPL}$$
$$0 = x^{\cup\cup}; 1' \cdot y \qquad \text{LPL}$$
$$0 = x^{\cup\cup} \cdot y.$$

We get $x^{\cup\cup} = x$ by first replacing y by $-x$, and then by $-x^{\cup\cup}$. Note that $1 = 1 + 1^{\cup}$, so by (6) and (8) we get (9) as follows: $1 = 1 + 1^{\cup} = 1^{\cup\cup} + 1^{\cup} = (1^{\cup} + 1)^{\cup} = 1^{\cup}$. For (10) we first get $-x \cdot x^{\cup\cup} = 0$ from $-x \cdot x = 0$ by (8), and then $(-x)^{\cup} \cdot x^{\cup} = 0$ by 1.11. Also, by (6) and (9), $(-x)^{\cup} + x^{\cup} = (-x + x)^{\cup} = 1^{\cup} = 1$. It follows that $(-x)^{\cup} = -x^{\cup}$. (11) follows from (6) and (10), and (12) follows from (6) and (8). For (13), we first note that the following formulas are equivalent:

$$0 = (x; y)^{\cup} \cdot z$$
$$0 = x; y \cdot z^{\cup} \qquad 1.11$$
$$0 = x^{\cup}; z^{\cup} \cdot y \qquad \text{LPL}$$
$$0 = y; z \cdot x^{\cup} \qquad \text{RPL}$$
$$0 = y^{\cup}; x^{\cup} \cdot z \qquad \text{LPL}.$$

Let $z = -(x; y)^{\cup}$, and then let $z = -(y^{\cup}; x^{\cup})$. It follows that $(x; y)^{\cup} = y^{\cup}; x^{\cup}$. By (13), (8), and the IL, we have $1'^{\cup} = 1'^{\cup}; 1'^{\cup\cup} = (1'^{\cup}; 1')^{\cup} = 1'^{\cup\cup} = 1'$, and hence, by

(10), $0'^\cup = (-1')^\cup = -1'^\cup = -1' = 0'$. Thus (14) holds. To prove (15) and (16), we assume $x, y \leq 1'$ and proceed as follows:

$$
\begin{aligned}
x;y &\leq x;1'\cdot 1';y && (3) \\
&= x \cdot y && \text{IL} \\
&= x;1'\cdot y && \text{IL} \\
&\leq x;\left(1'\cdot x^\cup ;y\right) && 1.4(1) \\
&\leq x;\left(1'^\cup ;y\right) && (3),(12) \\
&= x;y && (14),\text{IL}
\end{aligned}
$$

so (15) holds. We have

$$
\begin{aligned}
x &= 1';x \cdot 1' && \text{IL} \\
&\leq (1'\cdot 1';x^\cup);x && 1.4(2) \\
&\leq 1';x^\cup ;1' && (3) \\
&= x^\cup && \text{IL}
\end{aligned}
$$

so, by (12) and (8), we also get $x^\cup \leq x^{\cup\cup} = x$. Thus (16) holds, and the proof is complete.

COROLLARY 1.14. $RA \subseteq SA$.

PROOF. Use 1.13(5).

DEFINITION 1.15. For every relation-type algebra

$$
\mathfrak{A} = \left\langle A, +, \cdot, -, 0, 1, ;, \ ^\cup, 1' \right\rangle
$$

let

$$
\mathfrak{A}^\cup = \left\langle A, +, \cdot, -, 0, 1, ;^\cup, \ ^\cup, 1' \right\rangle
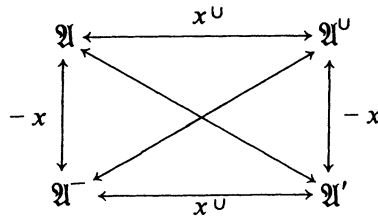$$

where $x;^\cup y = y;x$ for all $x, y \in A$,

$$
\mathfrak{A}^- = \left\langle A, \cdot, +, -, 1, 0, \ \dagger, \ ^\cup, 0' \right\rangle,
$$

and

$$
\mathfrak{A}' = \left\langle A, \cdot, +, -, 1, 0, \ \dagger^\cup, \ ^\cup, 0' \right\rangle,
$$

where $x \dagger^\cup y = y \dagger x$ for all $x, y \in A$, and let $x' = -x^\cup$ for every $x \in A$.

THEOREM 1.16. *Let* $\mathfrak{A} \in NA$. *Then* $\mathfrak{A} \cong \mathfrak{A}^\cup \cong \mathfrak{A}^- \cong \mathfrak{A}' = (\mathfrak{A}^-)^\cup = (\mathfrak{A}^\cup)^-$. *In particular, the operations* $^\cup$, $-$, *and* $'$ *are involutions (i.e.* $x = x^{\cup\cup} = --x = x''$) *and are isomorphisms between these algebras as indicated in the following diagram:*

PROOF. We get $x = x^{\cup\cup} = --x = x''$ from 1.13(8), (10), so $^{\cup}$, $-$, and $'$ are one-to-one and onto. The fact that they are also homomorphisms can be expressed by a number of identities, each of which can be easily derived from 1.13(6)–(11), (13), (14).

Given any theorem for NA's, we can obtain three other (not necessarily different) theorems for NA's by changing the operations in the original theorem to the corresponding operations of any one of the dual algebras $\mathfrak{A}^{\cup}$, $\mathfrak{A}^{-}$, and $\mathfrak{A}'$. (Thus there are three duality principles, which are discussed more fully in [1].) For example, if $\mathfrak{A} \in NA$, then $\mathfrak{A} \in WA$ iff $\mathfrak{A}$ satisfies any (or all) of the following identities: $(1' \cdot x); 1; 1 = (1' \cdot x); 1$, $1; (1; (1' \cdot x)) = 1; (1' \cdot x)$, $(0' + x) \dagger 0 \dagger 0 = (0' + x) \dagger 0$, $0 \dagger (0 \dagger (0' + x)) = 0 \dagger (0' + x)$.

2. Complex algebras.

DEFINITION 2.1. Let $\mathfrak{U} = \langle U, C, f, I \rangle$ where $C \subseteq {}^3U$, $f \in {}^UU$, and $I \subseteq U$. Then $\mathfrak{Cm}\,\mathfrak{U} = \langle \mathrm{Sb}\,U, \cup, \cap, {}_U\sim, \varnothing, U, ;, {}^{\cup}, I \rangle$ where $X; Y = \{z : \langle x, y, z \rangle \in C$ for some $x \in X, y \in Y\}$ and $X^{\cup} = \{fx : x \in X\}$ for all $X, Y \in \mathrm{Sb}\,U$. $\mathfrak{Cm}\,\mathfrak{U}$ is called the *complex algebra* of $\mathfrak{U}$.

The complex algebra of a structure is defined in Definition 3.8 of [6] for the case in which the structure has arbitrarily many finitary relations. If we were to treat f as a binary relation, then 2.1 would be a special case of the definition in [6].

The notion of complex algebra also occurs in universal algebra (see [2, p. 379]), but with a different meaning: the Boolean operations are not included, and the notion is defined only for algebras (structures with functions and no relations).

THEOREM 2.2. *Let* $\mathfrak{U} = \langle U, C, f, I \rangle$ *where* $C \subseteq {}^3U$, $f \in {}^UU$, *and* $I \subseteq U$.

(1) $\mathfrak{Cm}\,\mathfrak{U}$ *is a relation-type algebra,* $\mathfrak{Bl}\,\mathfrak{Cm}\,\mathfrak{U}$ *is a complete atomic BA, and the operations* ; *and* $^{\cup}$ *are normal and completely additive.*

(2) $\mathfrak{Cm}\,\mathfrak{U} \in NA$ *iff* $\mathfrak{U}$ *satisfies the following conditions for all* $x, y, z \in U$.

  (a) *If* $\langle x, y, z \rangle \in C$ *then* $\langle fx, z, y \rangle \in C$,

  (b) *if* $\langle x, y, z \rangle \in C$ *then* $\langle z, fy, x \rangle \in C$,

  (c) $x = y$ *iff there is some* $w \in I$ *such that* $\langle x, w, y \rangle \in C$.

(3) $\mathfrak{Cm}\,\mathfrak{U} \in WA$ *iff* $\mathfrak{U}$ *satisfies* (a)–(c) *and*

  (d) *if* $w \in I$ *and* $\langle w, x, x \rangle, \langle x, y, z \rangle \in C$, *then* $\langle w, z, z \rangle \in C$.

(4) $\mathfrak{Cm}\,\mathfrak{U} \in SA$ *iff* $\mathfrak{U}$ *satisfies* (a)–(c) *and*

  (e) *if* $\langle v, w, x \rangle, \langle x, y, z \rangle \in C$ *then there is some* $u \in U$ *such that* $\langle v, u, z \rangle \in C$.

(5) $\mathfrak{Cm}\,\mathfrak{U} \in RA$ *iff* $\mathfrak{U}$ *satisfies* (a)–(c) *and*

  (f) *if* $\langle v, w, x \rangle, \langle x, y, z \rangle \in C$ *then there is some* $u \in U$ *such that* $\langle v, u, z \rangle$, *and* $\langle w, y, u \rangle \in C$.

PROOF. (1) is easy to check. (See Theorem 3.3 of [6].)

(2) Assume $\mathfrak{Cm}\,\mathfrak{U} \in NA$. We first prove (a) and (b). Let $\langle x, y, z \rangle \in C$. Then $\{z\} \subseteq \{x\}; \{y\}$, or, equivalently, $\{x\}; \{y\} \cap \{z\} \neq \varnothing$. Applying the LPL yields $\{fx\}; \{z\} \cap \{y\} = \{x\}^{\cup}; \{z\} \cap \{y\} \neq \varnothing$, so $\langle fx, z, y \rangle \in C$. Similarly, the RPL yields $\langle z, fy, x \rangle \in C$.

If $w \in I$ and $\langle x, w, y \rangle \in C$, then $y \in \{x\}; \{w\} \subseteq \{x\}; I = \{x\}$ by the IL, so $x = y$. Conversely, if $x = y$ then $y \in \{x\} = \{x\}; I$, so there is some $w \in I$ such that $\langle x, w, y \rangle \in C$. Thus (c) holds.

To proceed further we need the following.

LEMMA. *If* $\mathfrak{U}$ *satisfies* (a) *and* (c), *then* $ffx = x$ *for every* $x \in U$.

PROOF. Let $x \in U$. Then $\langle x, w, x \rangle \in C$ for some $w \in I$ by (c). Using (a) twice we get $\langle fx, x, w \rangle \in C$ and then $\langle ffx, w, x \rangle \in C$. Consequently $ffx = x$ by (c).

Now we assume $\mathfrak{U}$ satisfies (a)–(c) and show $\mathfrak{Cm}\, \mathfrak{U} \in NA$. Since $\mathfrak{Bl}\, \mathfrak{Cm}\, \mathfrak{U} \in BA$ we need only prove the identity and Peircean laws. Let $X \subseteq U$. If $x \in X$, then $\langle x, w, x \rangle \in C$ for some $w \in I$ by (c), so $x \in \{x\}; \{w\} \subseteq X; I$. Thus $X \subseteq X; I$. On the other hand, if $y \in X; I$, then $\langle x, w, y \rangle \in C$ for some $x \in X$ and $w \in I$, but $x = y$ by (c), so $y \in X$. Thus $X; I \subseteq X$, and we have $X = X; I$. Let $x \in X$. Then $\langle fx, w, fx \rangle \in C$ for some $w \in I$ by (c). We get $\langle ffx, fx, w \rangle \in C$ by (a), and then $\langle w, x, x \rangle = \langle w, ffx, ffx \rangle \in C$ by the lemma and (b). Thus $x \in \{w\}; \{x\} \subseteq I; X$, so we have shown $X \subseteq I; X$. Now let $y \in I; X$, i.e. $\langle w, x, y \rangle \in C$ for some $w \in I$ and $x \in X$. Then $\langle y, fx, w \rangle \in C$ by (b), so $\langle fy, w, fx \rangle \in C$ by (a). Applying (c) we get $fy = fx$, so $y = ffy = ffx = x$ by the lemma. Thus $y \in X$, which shows $I; X \subseteq X$. The proof of the identity law is complete.

The Peircean laws hold in $\mathfrak{Cm}\, \mathfrak{U}$ iff the following formulas are equivalent for all $X, Y, Z \in \mathrm{Sb}\, U$: $X; Y \cap Z \neq \varnothing$, $X^{\cup}; Z \cap Y \neq \varnothing$, $Z; Y^{\cup} \cap X \neq \varnothing$. We shall only prove the first two are equivalent; the other equivalences can be proved similarly.

Let $X; Y \cap Z \neq \varnothing$. Then $\langle x, y, z \rangle \in C$ for some $x \in X$, $y \in Y$, $z \in Z$. By (a), $\langle fx, z, y \rangle \in C$, so $\varnothing \neq \{fx\}; \{z\} \cap \{y\} = \{x\}^{\cup}; \{z\} \cap \{y\} \subseteq X^{\cup}; Z \cap Y$. Now suppose $X^{\cup}; Z \cap Y \neq \varnothing$. Then there are $x \in X$, $y \in Y$, $z \in Z$ such that $\langle fx, z, y \rangle \in C$. By the lemma and (a), $\langle x, y, z \rangle = \langle ffx, y, z \rangle \in C$, so $X; Y \cap Z \neq \varnothing$.

(3) Assume $\mathfrak{Cm}\, \mathfrak{U} \in WA$. Then (a)–(c) hold by (2). Suppose $w \in I$ and $\langle w, x, x \rangle$, $\langle x, y, z \rangle \in C$. Then $\{x\} \subseteq \{w\}; \{x\}$ and $z \in \{x\}; \{y\}$, so $z \in \{w\}; \{x\}; \{y\} \subseteq \{w\}; U; U = \{w\}; U$ by the WL. Consequently $\langle w, v, z \rangle \in C$ for some $v \in U$. But $v = z$ since $z \in \{w\}; \{v\} \subseteq I; \{v\} = \{v\}$ by the IL, so $\langle w, z, z \rangle \in C$. Thus (d) holds.

Assume $\mathfrak{U}$ satisfies (a)–(d). Then $\mathfrak{Cm}\, \mathfrak{U} \in NA$ by (2), so we need only prove the WL for $\mathfrak{Cm}\, \mathfrak{U}$. Let $W \in \mathrm{Sb}\, U$ and $z \in (I \cap W); U; U$. Then there are $w \in I \cap W$, $u \in U$, $x \in (I \cap W); U$, and $y \in U$ such that $\langle w, u, x \rangle$, $\langle x, y, z \rangle \in C$. But $x \in \{w\}; \{u\} \subseteq I; \{u\} = \{u\}$, so $x = u$. It follows from (d) that $\langle w, z, z \rangle \in C$, so $z \in \{w\}; \{z\} \subseteq (I \cap W); U$. Thus $(I \cap W); U; U \subseteq (I \cap W); U$. The opposite inclusion follows from 1.13(4).

(4) Assume $\mathfrak{Cm}\, \mathfrak{U} \in SA$. Then (a)–(c) hold by (2). Suppose $\langle v, w, x \rangle$, $\langle x, y, z \rangle \in C$. Then $z \in \{x\}; \{y\} \subseteq \{v\}; \{w\}; \{y\} \subseteq \{v\}; U; U = \{v\}; U$ by the SL, so $\langle v, u, z \rangle \in C$ for some $u \in U$. Thus (e) holds.

Now suppose $\mathfrak{U}$ satisfies (a)–(c) and (e). Then $\mathfrak{Cm}\, \mathfrak{U} \in NA$ by (2), and $V; U \subseteq V; U; U$ for every $V \in \mathrm{Sb}\, U$ by 1.13(4). Suppose $z \in V; U; U$. Then there are $v \in V$, $w \in U$, $x \in V; U$, and $y \in U$ such that $\langle v, w, x \rangle$, $\langle x, y, z \rangle \in C$. By (e) there is some $u \in U$ such that $\langle v, u, z \rangle \in C$, so $z \in \{v\}; \{u\} \subseteq V; U$. Therefore $V; U; U = V; U$, and $\mathfrak{Cm}\, \mathfrak{U} \in SA$.

(5) Assume $\mathfrak{Cm}\, \mathfrak{U} \in RA$. Then (a)–(c) hold by (2). Let $\langle v, w, x \rangle$, $\langle x, y, z \rangle \in C$. Then $z \in \{x\}; \{y\} \subseteq \{v\}; \{w\}; \{y\} = \{v\}; (\{w\}; \{y\})$ by the AL, so there is some $u \in \{w\}; \{y\}$ (hence $\langle w, y, u \rangle \in C$) such that $\langle v, u, z \rangle \in C$. Thus (f) holds.

Suppose $\mathfrak{U}$ satisfies (a)–(c) and (f). Then $\mathfrak{C}\mathfrak{m}\,\mathfrak{U}\in NA$ by (2). We need only show $V;W;Y\subseteq V;(W;Y)$ for all $V,W,Y\in \mathrm{Sb}\,U$, since the AL follows from this by 1.16. Let $z\in V;W;Y$. Then there are $v\in V$, $w\in W$, $x\in V;W$, and $y\in Y$ such that $\langle v, w, x\rangle$, $\langle x, y, z\rangle\in C$. By (f) there is some $u\in U$ such that $\langle v, u, z\rangle$, $\langle w, y, u\rangle\in C$. Hence $z\in \{v\};\{u\}\subseteq \{v\};(\{w\};\{y\})\subseteq V;(W;Y)$. Thus $V;W;Y\subseteq V;(W;Y)$.

Problem 2.3. Is there a set of first-order conditions on $\mathfrak{U}=\langle U, C, f, I\rangle$ which is equivalent to $\mathfrak{C}\mathfrak{m}\,\mathfrak{U}\in RRA$? In other words is $\{\mathfrak{U}:\mathfrak{C}\mathfrak{m}\,\mathfrak{U}\in RRA\}$ an elementary class?

Some sufficient conditions for $\mathfrak{C}\mathfrak{m}\,\mathfrak{U}\in RRA$ are known. For example, it follows from Theorems 4.27 and 4.29 of [7] that $\mathfrak{C}\mathfrak{m}\,\mathfrak{U}\in RRA$ whenever $\mathfrak{U}$ satisfies (a)–(c) and (f), and $I=\{w:\langle x, w, x\rangle\in C \text{ for some } x\in U\}$. An alternative formulation of this condition can be obtained by showing that if $\mathfrak{U}$ satisfies (a)–(c) and (f) (i.e. $\mathfrak{C}\mathfrak{m}\,\mathfrak{U}\in RA$) then $I=\{w:\langle x, w, x\rangle\in C \text{ for some } x\in U\}$ iff C is functional (i.e. if $\langle x, y, z\rangle$, $\langle x, y, w\rangle\in C$, then $z=w$). A more general sufficient (but not necessary) condition for $\mathfrak{C}\mathfrak{m}\,\mathfrak{U}\in RRA$ can be obtained from Theorem 7 of [11]: if $\mathfrak{U}$ satisfies (a)–(c) and (f), and for every $u\in U$ there are $p, q\in U$ such that $\langle p, u, q\rangle\in C$, and $w\in I$ whenever $\langle p, w, p\rangle\in C$ or $\langle q, w, q\rangle\in C$, then $\mathfrak{C}\mathfrak{m}\,\mathfrak{U}\in RRA$.

DEFINITION 2.4. Let α be a nonzero cardinal and let $X\subseteq \{1, 2, 3\}$. Then

$$D_\alpha = \bigcup_{\kappa<\alpha} \{\langle 0, \kappa, \kappa\rangle, \langle \kappa, 0, \kappa\rangle, \langle \kappa, \kappa, 0\rangle\},$$

$$C_\alpha(X) = D_\alpha \cup \{\langle \kappa, \lambda, \mu\rangle : 0 < \kappa, \lambda, \mu < \alpha \text{ and } |\{\kappa, \lambda, \mu\}|\in X\},$$

and

$$\mathfrak{E}_\alpha(X) = \mathfrak{C}\mathfrak{m}\,\langle \alpha, C_\alpha(X), \mathrm{Id}_\alpha, \{0\}\rangle.$$

THEOREM 2.5. (1) *Let* $\mathfrak{U}\in NA$. *Then* $\mathfrak{U}\cong \mathfrak{E}_\alpha(X)$ *for some* $\alpha > 0$ *and some* $X\subseteq \{1, 2, 3\}$ *iff the following conditions hold:*

   (a) $x^{\cup} = x$ *for every* $x\in A$,
   (b) $\mathfrak{U}$ *is complete and atomic,*
   (c) 1' *is an atom of* $\mathfrak{A}$,
   (d) *if* f *is a permutation of the atoms of* $\mathfrak{A}$, *and* $f1' = 1'$, *then* f *can be extended to an automorphism of* $\mathfrak{A}$.

(2) $\mathfrak{E}_\alpha(X) = \mathfrak{E}_\beta(Y)$ *iff one of the following conditions holds:*
   (a) $1 = \alpha = \beta$,
   (b) $2 = \alpha = \beta$ *and* $\{1\}\cap X = \{1\}\cap Y$,
   (c) $3 = \alpha = \beta$ *and* $\{1, 2\}\cap X = \{1, 2\}\cap Y$,
   (d) $3 < \alpha = \beta$ *and* $X = Y$.

(3) $\mathfrak{E}_\alpha(X)\in WA \sim SA$ *iff one of the following conditions holds:*
   (a) $3 = \alpha$ *and* X *is* $\{3\}$ *or* $\{1, 3\}$,
   (b) $3 < \alpha$ *and* X *is* $\varnothing$, $\{1\}$, $\{3\}$, *or* $\{1, 3\}$.

(4) $\mathfrak{E}_\alpha(X)\in SA \sim RA$ *iff one of the following conditions holds:*
   (a) $4 = \alpha$ *and* $X = \{1, 3\}$,
   (b) $4 < \alpha$ *and* $X = \{3\}$.

(5) $\mathfrak{E}_\alpha(X)\in RA$ *in all cases not listed in* (3) *or* (4).

PROOF. (1) and (2) are fairly obvious. For (3)–(5) use 2.2.

COROLLARY 2.6. $RA \subset SA \subset WA$.

Algebras which show $WA \subset NA$ will be constructed later. It is well known that $RRA \subset RA$. (This was first proved in [8].) For some values of α and X, $\mathfrak{E}_\alpha(X)$ is a nonrepresentable RA. For example, if $4 \leqslant \alpha$, then $\mathfrak{E}_\alpha(\{2\}) \in RA \sim RRA$ (see [10] or [12, p. 63]), and it follows from [9] that $\mathfrak{E}_\alpha(\{1,3\}) \in RA \sim RRA$ iff there is no projective plane of order $\alpha - 2$. The Bruck-Ryser Theorem, on the nonexistence of projective planes, implies that $\mathfrak{E}_\alpha(\{1,3\}) \in RA \sim RRA$ for infinitely many values of α. These nonrepresentable RA's are used in [14] to show that RRA is not finitely axiomatizable. In the other direction, we have $\mathfrak{E}_\alpha(X) \in RRA$ whenever $\mathfrak{E}_\alpha(X) \in RA$ and $\alpha \leqslant 3$, $\mathfrak{E}_4(\{3\}) \in RRA$, and $\mathfrak{E}_\alpha(\{1,2\})$, $\mathfrak{E}_\alpha(\{1,2,3\}) \in RRA$ for all α. (See [12].) The remaining unsettled cases are collected in the following problem.

Problem 2.7. For what values of α is $\mathfrak{E}_\alpha(\{2,3\}) \in RRA$ or $\mathfrak{E}_\alpha(\{1,3\}) \in RRA$?

3. Atom structures.

DEFINITION 3.1. Suppose $\mathfrak{A}$ is a relation-type algebra such that $\mathfrak{Bl}\,\mathfrak{A} \in BA$. Let At $\mathfrak{A}$ be the set of atoms of $\mathfrak{A}$, i.e. At $\mathfrak{A} = \{x: 0 \neq x \in A$ and for every $y \in A$, either $x \cdot y = 0$ or $x \cdot y = x\}$. We say that $^{\cup}$ *preserves atoms* in $\mathfrak{A}$ if $x^{\cup} \in$ At $\mathfrak{A}$ whenever $x \in$ At $\mathfrak{A}$.

DEFINITION 3.2. Suppose $\mathfrak{A}$ is a relation-type algebra, $\mathfrak{Bl}\,\mathfrak{A} \in BA$, and $^{\cup}$ preserves atoms in $\mathfrak{A}$. Then $\mathfrak{At}\,\mathfrak{A} = \langle$At $\mathfrak{A}, C^{(\mathfrak{A})}, {}^{\cup}, I^{(\mathfrak{A})}\rangle$ is the *atom structure of* $\mathfrak{A}$, where $C^{(\mathfrak{A})} = \{\langle x,y,z\rangle: x,y,z \in$ At $\mathfrak{A}$ and $x; y \geqslant z\}$ and $I^{(\mathfrak{A})} = \{x: 1^{,} \geqslant x \in$ At $\mathfrak{A}\}$. The triples in $C^{(\mathfrak{A})}$ are the *cycles of* $\mathfrak{A}$, and the elements of $I^{(\mathfrak{A})}$ are the *identity atoms* of $\mathfrak{A}$.

THEOREM 3.3. *Let* $\mathfrak{U} = \langle U, C, f, I\rangle$ *where* $C \subseteq {}^3U$, $f \in {}^UU$, *and* $I \subseteq U$. *Then* $^{\cup}$ *preserves atoms in* $\mathfrak{Cm}\,\mathfrak{U}$ *and* $\mathfrak{At}\,\mathfrak{Cm}\,\mathfrak{U} \cong \mathfrak{U}$. *In fact,* $\langle \{x\}: x \in U\rangle$ *is an isomorphism from* $\mathfrak{U}$ *onto* $\mathfrak{At}\,\mathfrak{Cm}\,\mathfrak{U}$.

THEOREM 3.4. *If* $\mathfrak{A} \in NA$, *then* $^{\cup}$ *preserves atoms in* $\mathfrak{A}$.

PROOF. Let $x \in$ At $\mathfrak{A}$. Then for every $y \in A$, either $x \cdot y^{\cup} = 0$ or $x \cdot y^{\cup} = x$. But $x \cdot y^{\cup} = 0$ iff $x^{\cup} \cdot y = 0$ by 1.11, and $x \cdot y^{\cup} = x$ iff $x^{\cup} \cdot y = x^{\cup}$ by 1.13(11), (8). Hence $x^{\cup} \cdot y = 0$ or $x^{\cup} \cdot y = x^{\cup}$. Also, $0 \neq x^{\cup}$ since otherwise $0 = 0^{\cup} = x^{\cup\cup} = x$ by 1.13(7), (8). Thus $x^{\cup} \in$ At $\mathfrak{A}$.

The notion of atom structure was introduced as a way of recapturing the structure from which a complex algebra is constructed, as in 3.3. (A similar situation occurs in [4, pp. 453–461].) But this construction can be applied to any NA, as shown by 3.4, so we may consider the algebra $\mathfrak{Cm}\,\mathfrak{At}\,\mathfrak{A}$ obtained from an arbitrary $\mathfrak{A} \in NA$. We shall see that $\mathfrak{Cm}\,\mathfrak{At}\,\mathfrak{A} \cong \mathfrak{A}$ whenever $\mathfrak{A}$ is complete and atomic, but $\mathfrak{Cm}\,\mathfrak{At}\,\mathfrak{A} \notin NA$ for some complete nonatomic $\mathfrak{A} \in NA$. On the other hand, $\mathfrak{Cm}\,\mathfrak{At}\,\mathfrak{A} \in WA$ whenever $\mathfrak{A} \in WA$. To prove the latter result we need a theorem about atoms in WA's which fails for NA's.

THEOREM 3.5. *Let* $\mathfrak{A} \in WA$. *If* $x \in$ At $\mathfrak{A}$, *then* $x; x^{\cup} \cdot 1^{,}$, $x^{\cup}; x \cdot 1^{,} \in$ At $\mathfrak{A}$.

PROOF. Let $x \in \text{At } \mathfrak{A}$. We first show $x; x^{\cup} \cdot 1' \in \text{At } \mathfrak{A}$. We have $0 \neq x = 1'; x \cdot x$ by the IL, so $0 \neq x; x^{\cup} \cdot 1'$ by the RPL. Let $y \in A$, and suppose $x; x^{\cup} \cdot 1' \cdot y \neq 0$. Then $(1' \cdot y); x \cdot x \neq 0$ by the RPL, and hence $x \leqslant (1' \cdot y); x$. Consequently

$$
\begin{aligned}
x; x^{\cup} \cdot 1' &\leqslant (1' \cdot y); x; x^{\cup} \cdot 1' & &1.13(3) \\
&\leqslant (1' \cdot y); 1; 1 \cdot 1' & &1.13(3) \\
&= (1' \cdot y); 1 \cdot 1' & &\text{WL} \\
&\leqslant (1' \cdot y); \left(1 \cdot (1' \cdot y)^{\cup}; 1'\right) & &1.4(1) \\
&\leqslant y; (1'^{\cup}; 1') & &1.13(3), (12) \\
&= y & &1.13(14), \text{IL}
\end{aligned}
$$

so $x; x^{\cup} \cdot 1' \cdot y = x; x^{\cup} \cdot 1'$. Thus $x; x^{\cup} \cdot 1' \in \text{At } \mathfrak{A}$. One can prove $x^{\cup}; x \cdot 1' \in \text{At } \mathfrak{A}$ similarly, or notice that $x^{\cup}; x \cdot 1' = (x; x^{\cup} \cdot 1')^{\cup}$ by 1.13(8), (11), (13), (16) and use 3.4, or apply 1.16.

DEFINITION 3.6. $\mathfrak{R} = \mathfrak{C}\mathfrak{m}\langle\{0, 1, 2\}, C, \text{Id}_3, 1'\rangle$ where $1' = \{0, 1\}$ and $C = \{\langle 0, 0, 0\rangle, \langle 1, 1, 1\rangle, \langle 0, 2, 2\rangle, \langle 2, 0, 2\rangle, \langle 2, 2, 0\rangle, \langle 1, 2, 2\rangle, \langle 2, 1, 2\rangle, \langle 2, 2, 1\rangle\}$.

THEOREM 3.7. (1) $\mathfrak{R} \in NA \sim WA$ and $WA \subset NA$.
(2) *Theorem 3.5 fails in* $\mathfrak{R}$.

PROOF. It is easy to check that $\mathfrak{R}$ is the complex algebra of a structure satisfying 2.2(a)–(c), so $\mathfrak{R} \in NA$. $\mathfrak{R} \notin WA$ since $\{0\}; \{0, 1, 2\} = \{0, 2\}$ but $\{0\}; \{0, 1, 2\};$ $\{0, 2\}; \{0, 1, 2\} = \{0, 1, 2\}$. Theorem 3.5 fails since $\{2\} \in \text{At } \mathfrak{R}$ but $\{2\};$ $\{2\}^{\cup} \cap 1' = 1' \notin \text{At } \mathfrak{R}$.

THEOREM 3.8. *If* $\mathfrak{A} \in WA$, *then* $\mathfrak{C}\mathfrak{m} \mathfrak{A}\mathfrak{t} \mathfrak{A} \in WA$.

PROOF. It suffices to show $\mathfrak{A}\mathfrak{t} \mathfrak{A}$ satisfies 2.2(a)–(d). Let $\langle x, y, z\rangle \in C^{(\mathfrak{A})}$. Then $x; y \geqslant z$, so $x; y \cdot z \neq 0$. We get $x^{\cup}; z \cdot y \neq 0$ and $z; y^{\cup} \cdot x \neq 0$ by the Peircean laws, so $x^{\cup}; z \geqslant y$ and $z; y^{\cup} \geqslant x$ since $x, y \in \text{At } \mathfrak{A}$. Hence $\langle x^{\cup}, z, y\rangle, \langle z, y^{\cup}, x\rangle \in C^{(\mathfrak{A})}$. Thus $\mathfrak{A}\mathfrak{t} \mathfrak{A}$ satisfies 2.2(a), (b).

If $w \in I^{(\mathfrak{A})}$ and $\langle x, w, y\rangle \in C^{(\mathfrak{A})}$, then $y \leqslant x; w \leqslant x; 1' = x$, so $y = x$ since x, $y \in \text{At } \mathfrak{A}$. On the other hand, for any $x \in \text{At } \mathfrak{A}$ we have $x^{\cup}; x \cdot 1' \in \text{At } \mathfrak{A}$ by 3.5, and $\langle x, x^{\cup}; x \cdot 1', x\rangle \in C^{(\mathfrak{A})}$ since $x = x; 1' \cdot x \leqslant x; (1' \cdot x^{\cup}; x)$ by the IL and 1.4(1). Thus 2.2(c) holds.

Suppose $w \in I^{(\mathfrak{A})}$ and $\langle w, x, x\rangle, \langle x, y, z\rangle \in C^{(\mathfrak{A})}$. Then $w \leqslant 1'$, $x \leqslant w; x$, and $z \leqslant x; y$, so $z \leqslant x; y \leqslant w; x; y \leqslant w; 1; 1 = w; 1$ by 1.13(3) and the WL. Consequently $z = w; 1 \cdot z \leqslant w; (1 \cdot w^{\cup}; z) \leqslant w; (1'^{\cup}; z) = w; z$ by 1.4(1), 1.13(3), (12), (14), and the IL. Thus $\langle w, z, z\rangle \in C^{(\mathfrak{A})}$, so $\mathfrak{A}\mathfrak{t} \mathfrak{A}$ satisfies 2.2(d).

THEOREM 3.9. *There is a complete* $\mathfrak{A} \in NA$ *such that* $\mathfrak{C}\mathfrak{m} \mathfrak{A}\mathfrak{t} \mathfrak{A} \notin NA$.

PROOF. Let $\mathfrak{B} = \langle B, +, \cdot, -, 0, 1\rangle$ be the direct product of a complete atomless BA (e.g. §20(C) of [16]) and a 2-element BA. Then $\mathfrak{B}$ is a complete infinite BA with

exactly one atom. Let 0' be the unique atom of $\mathfrak{B}$ and let $1' = -0'$. Define ; on $B \times B$ as follows:

$$x;y = \begin{cases} 0 & \text{if } x = 0 \text{ or } y = 0, \\ x \cdot y & \text{if } 0 \neq x \leqslant 1' \text{ and } 0 \neq y \leqslant 1', \\ x \cdot y + 0' & \text{if } x \geqslant 0' \text{ and } 0 \neq y \leqslant 1', \\ x \cdot y + 0' & \text{if } 0 \neq x \leqslant 1' \text{ and } y \geqslant 0', \\ 1 & \text{if } x \geqslant 0' \text{ and } y \geqslant 0'. \end{cases}$$

Set $\mathfrak{A} = \langle B, +, \cdot, -, 0, 1, ;, \mathrm{Id}_B, 1' \rangle$. Notice that ; is commutative. We shall see that $\mathfrak{A} \in NA$ but the IL fails in $\mathfrak{Cm}\,\mathfrak{At}\,\mathfrak{A}$.

The verification of the IL in $\mathfrak{A}$ splits into three cases: if $x = 0$, then $x;1' = 1';x = 0 = x$; if $0 \neq x \leqslant 1'$, then $x;1' = 1';x = 1' \cdot x = x$; finally, if $x \geqslant 0'$, then $x;1' = 1';x = x \cdot 1' + 0' = x \cdot 1' + x \cdot 0' = x \cdot (1' + 0') = x \cdot 1 = x$.

Let $x, y, z \in B$ and set $a = x; y \cdot z$, $b = x; z \cdot y$, and $c = z; y \cdot x$. To show $\mathfrak{A}$ satisfies the Peircean laws it suffices, since converse is the identity in $\mathfrak{A}$, to show $a = 0$ iff $b = 0$ iff $c = 0$. If $0 \in \{x, y, z\}$, then $a = b = c = 0$, so assume $0 \neq x, y, z$. The following facts are easy to check. If $x + y \leqslant 1'$ or $x + z \leqslant 1'$ or $y + z \leqslant 1'$, then $a = b = c$. If $0' \leqslant x \cdot y$ and $z \leqslant 1'$, then $a = z \neq 0$ and $0' \leqslant b = c$. Similarly for permutations of x, y, and z. Finally, if $0' \leqslant x \cdot y \cdot z$, then $a = x \neq 0$, $b = y \neq 0$, and $c = z \neq 0$. This exhausts the possibilities, and in each case $a = 0$ iff $b = 0$ iff $c = 0$. Thus $\mathfrak{A} \in NA$.

However, $\mathrm{At}\,\mathfrak{A} = \{0'\}$, $C^{(\mathfrak{A})} = \{\langle 0', 0', 0' \rangle\}$, and $I^{(\mathfrak{A})} = \varnothing$, so $\mathfrak{Cm}\,\mathfrak{At}\,\mathfrak{A} \cong \langle \{0, 1\}, +, \cdot, -, 0, 1, \cdot, \mathrm{Id}_{\{0,1\}}, 0 \rangle$. The IL fails since $1;0 = 1 \cdot 0 = 0 \neq 1$, so $\mathfrak{Cm}\,\mathfrak{At}\,\mathfrak{A} \notin NA$.

Theorem 3.9 shows that the hypothesis of 3.8 cannot be weakened from $\mathfrak{A} \in WA$ to $\mathfrak{A} \in NA$. The next theorem shows that no stronger conclusion than $\mathfrak{Cm}\,\mathfrak{At}\,\mathfrak{A} \in WA$ can be obtained by assuming $\mathfrak{A} \in RRA$.

THEOREM 3.10. *There is some* $\mathfrak{A} \in RRA$ *such that* $\mathfrak{Cm}\,\mathfrak{At}\,\mathfrak{A} \in WA \sim SA$.

PROOF. Let $U = \{e_{00}, e_{11}, e_{22}, d_{11}, d_{22}, d_{01}, d_{10}, d_{02}, d_{20}\}$, $U' = U \cup \{d_{12}^r : r \in Q\} \cup \{d_{21}^r : r \in Q\}$ (where Q is the set of rational numbers), and $I = \{e_{00}, e_{11}, e_{22}\}$. Define f on U' as follows, for all $r \in Q$:

x	e_{00}	e_{11}	e_{22}	d_{11}	d_{22}	d_{01}	d_{10}	d_{02}	d_{20}	d_{12}^r	d_{21}^r
fx	e_{00}	e_{11}	e_{22}	d_{11}	d_{22}	d_{10}	d_{01}	d_{20}	d_{02}	d_{21}^r	d_{12}^r

Notice that U is closed under f. If $\langle x, y, z \rangle \in {}^3 U'$ then the *transforms* of $\langle x, y, z \rangle$ are $\langle x, y, z \rangle$, $\langle fx, z, y \rangle$, $\langle y, fz, fx \rangle$, $\langle fy, fx, fz \rangle$, $\langle fz, x, fy \rangle$, and $\langle z, fy, x \rangle$. Let C' be the set of transforms of the following triples, for all $r, s \in Q$:

$$\langle e_{11}, e_{11}, e_{11} \rangle, \quad \langle e_{22}, e_{22}, e_{22} \rangle, \quad \langle e_{00}, e_{00}, e_{00} \rangle,$$
$$\langle d_{11}, e_{11}, d_{11} \rangle, \quad \langle d_{22}, e_{22}, d_{22} \rangle, \quad \langle d_{10}, e_{00}, d_{10} \rangle,$$
$$\langle d_{01}, e_{11}, d_{01} \rangle, \quad \langle d_{02}, e_{22}, d_{02} \rangle, \quad \langle d_{20}, e_{00}, d_{20} \rangle,$$
$$\langle d_{21}^r, e_{11}, d_{21}^r \rangle, \quad \langle d_{12}^r, e_{22}, d_{12}^r \rangle,$$

$$\langle d_{11}, d_{11}, d_{11}\rangle, \quad \langle d_{22}, d_{22}, d_{22}\rangle, \quad \langle d_{01}, d_{12}^r, d_{02}\rangle,$$
$$\langle d_{01}, d_{11}, d_{01}\rangle, \quad \langle d_{02}, d_{22}, d_{02}\rangle,$$
$$\langle d_{21}^r, d_{11}, d_{21}^s\rangle, \quad \langle d_{12}^r, d_{22}, d_{12}^s\rangle.$$

Let $C = C' \cap^3 U$. Let $\mathfrak{B} = \mathfrak{Cm}\langle U, C, f, I\rangle$ and $\mathfrak{C} = \mathfrak{Cm}\langle U', C', f, I\rangle$. We will show:

(1) $\mathfrak{B} \in WA \sim SA$,

(2) $\mathfrak{C} \in RRA$,

(3) for some $\mathfrak{A} \subseteq \mathfrak{C}$, $\mathfrak{Cm}\,\mathfrak{At}\,\mathfrak{A} \cong \mathfrak{B}$.

If (1)–(3) hold, then the proof is complete, since $\mathfrak{A} \in RRA$ by (2) and (3), but $\mathfrak{Cm}\,\mathfrak{At}\,\mathfrak{A} \in WA \sim SA$ by (1) and (3).

It is clear that $\langle U, C, f, I\rangle$ satisfies 2.2(a)–(c), and it is easy to show that it also satisfies 2.2(d), so $\mathfrak{B} \in WA$. On the other hand, $\mathfrak{B} \notin SA$ since $\langle U, C, f, I\rangle$ does not satisfy 2.2(e): $\langle d_{01}, d_{10}, e_{00}\rangle, \langle e_{00}, d_{02}, d_{02}\rangle \in C$ but there is no $x \in U$ such that $\langle d_{01}, x, d_{02}\rangle \in C$.

To show $\mathfrak{C} \in RRA$ we need two countable infinite disjoint sets K and L, an element $m \notin K \cup L$, and a function $p: (K \times L) \cup (L \times K) \to Q$ which has the following properties:

(4) $p(k, l) = p(l, k)$ for every $k \in K$ and $l \in L$,

(5) if $k_1, k_2 \in K$, $k_1 \neq k_2$, and $r, s \in Q$, then there is some $l \in L$ such that $p(k_1, l) = r$ and $p(k_2, l) = s$,

(6) if $l_1, l_2 \in L$, $l_1 \neq l_2$, and $r, s \in Q$, then there is some $k \in K$ such that $p(k, l_1) = r$ and $p(k, l_2) = s$.

Such sets and functions are easy to construct. For example, let $M_0 = Q$, $M_1 = M_0 \times Q \times M_0 \times Q$, and $M_{n+2} = M_n \cup (M_{n+1} \times Q \times M_{n+1} \times Q)$ for every $n < \omega$. Set $K = \bigcup_{n<\omega} M_{2n}$ and $L = \bigcup_{n<\omega} M_{2n+1}$. If $\langle x, r, y, s\rangle \in M_n \times Q \times M_n \times Q$ and $x \neq y$, then set

(7) $p(x, \langle x, r, y, s\rangle) = p(\langle x, r, y, s\rangle, x) = r$,

(8) $p(y, \langle x, r, y, s\rangle) = p(\langle x, r, y, s\rangle, y) = s$,

and let $p(k, l) = p(l, k) = 0$ for all $k \in K$, $l \in L$, not covered in (7) or (8). (In fact, since (7) and (8) imply (5) and (6), p can be extended to $(K \times L) \cup (L \times K)$ in any manner consistent with (4), (7), and (8).)

Suppose K, L, m, and p satisfy (4)–(6). Then we get an embedding F of $\mathfrak{C}$ into $\mathfrak{Re}(K \cup L \cup \{m\})$ as follows:

$$F(e_{00}) = \{\langle m, m\rangle\}, \quad F(d_{01}) = \{m\} \times K, \quad F(e_{11}) = \mathrm{Id}_K,$$
$$F(d_{10}) = K \times \{m\}, \quad F(e_{22}) = \mathrm{Id}_L, \qquad F(d_{02}) = \{m\} \times L,$$
$$F(d_{11}) = K \times K, \qquad F(d_{20}) = L \times \{m\}, \quad F(d_{22}) = L \times L,$$
$$F(d_{12}^r) = \{\langle k, l\rangle : k \in K, l \in L, p(k, l) = r\} \quad \text{for all } r \in Q,$$
$$F(d_{21}^r) = \{\langle l, k\rangle : l \in L, k \in K, p(l, k) = r\} \quad \text{for all } r \in Q,$$
$$F(X) = \bigcup_{x \in X} F(x) \quad \text{for all } X \subseteq U'.$$

Thus $\mathfrak{C} \in RRA$. Now we choose $\mathfrak{A} \subseteq \mathfrak{C}$. Let H be the set of all subsets of Q which are finite unions of intervals $Q \cap (a, b)$ where a and b are irrational. Then $\langle H, \cup,$

$\cap, _Q \sim, \varnothing, Q\rangle$ is an atomless *BA*. Let A be the set of subsets of U' of the form $X \cup \{d_{12}^r : r \in R\} \cup \{d_{21}^s : s \in S\}$ where $X \subseteq U$ and $R, S \in H$. Then it is not difficult to verify that A is closed under $\cup, \cap, _{U'} \sim, ;$, and $^\cup$, hence is the universe of a subalgebra $\mathfrak{A}$ of $\mathfrak{C}$, and that $\mathfrak{C} \mathrm{m} \mathfrak{A} \mathrm{t} \mathfrak{A} \cong \mathfrak{B}$.

Problem 3.11. Characterize the class $\{\mathfrak{C} \mathrm{m} \mathfrak{A} \mathrm{t} \mathfrak{A} : \mathfrak{A} \in RRA\}$. Does it contain every complete atomic *WA*? What variety does it generate?

Problem 3.12 (*raised by Don Pigozzi*). Suppose $K \subseteq NA$, K is a variety, and $\mathfrak{C} \mathrm{m} \mathfrak{A} \mathrm{t} \mathfrak{A} \in K$ for every $\mathfrak{A} \in K$. Is $K \subseteq WA$?

The algebras used in the proofs of 3.9 and 3.10 are not atomic. The next theorem shows that this is unavoidable.

THEOREM 3.13. *Assume* $\mathfrak{A} \in NA$ *and* $\mathfrak{A}$ *is atomic.*

(1) $\mathfrak{A}$ *is isomorphic to a subalgebra of* $\mathfrak{C} \mathrm{m} \mathfrak{A} \mathrm{t} \mathfrak{A}$.

(2) $\mathfrak{A} \cong \mathfrak{C} \mathrm{m} \mathfrak{A} \mathrm{t} \mathfrak{A}$ *iff* $\mathfrak{A}$ *is complete.*

(3) $\mathfrak{C} \mathrm{m} \mathfrak{A} \mathrm{t} \mathfrak{A} \in NA$.

(4) $\mathfrak{C} \mathrm{m} \mathfrak{A} \mathrm{t} \mathfrak{A}$ *is a WA, SA, RA, or RRA iff* $\mathfrak{A}$ *is a WA, SA, RA, or RRA, respectively.*

PROOF. (1) Define $f : A \to \mathrm{Sb} \, \mathrm{At} \, \mathfrak{A}$ by $fx = \{a : x \geqslant a \in \mathrm{At} \, \mathfrak{A}\}$ for every $x \in A$. It is well known that f embeds $\mathfrak{Bl} \, \mathfrak{A}$ into $\mathfrak{Bl} \, \mathfrak{C} \mathrm{m} \mathfrak{A} \mathrm{t} \mathfrak{A}$ (see [16, Theorem 9.1, p. 28]), and $x = \Sigma fx$ for every $x \in A$. Obviously, $f1' = I^{(\mathfrak{A})}$, so we need only show $f(x; y) = fx; fy$ and $f(x^\cup) = (fx)^\cup$ for all $x, y \in A$. To establish the first formula, note that the following formulas are equivalent for every $a \in \mathrm{At} \, \mathfrak{A}$.

$$a \in f(z; y),$$
$$a \leqslant x; y = \Sigma \{b; c : b \in fx, c \in fy\},$$
$$a \leqslant b; c \quad \text{for some } b \in fx, c \in fy,$$
$$\langle b, c, a \rangle \in C^{(\mathfrak{A})} \quad \text{for some } b \in fx, c \in fy,$$
$$a \in fx; fy.$$

Another sequence of equivalent formulas, with $a \in \mathrm{At} \, \mathfrak{A}$, proves $f(x^\cup) = (fx)^\cup$;

$$a \in f(x^\cup), \qquad a \leqslant x^\cup = (\Sigma fx)^\cup = \sum_{b \in fx} b^\cup,$$
$$a = b^\cup \quad \text{for some } b \in fx, \qquad a \in (fx)^\cup.$$

(2) If $\mathfrak{A} \cong \mathfrak{C} \mathrm{m} \mathfrak{A} \mathrm{t} \mathfrak{A}$, then $\mathfrak{A}$ is obviously complete. Suppose $\mathfrak{A}$ is complete. In view of (1), it suffices to show f is onto. Let $X \subseteq \mathrm{At} \, \mathfrak{A}$. Then ΣX exists and $f(\Sigma X) = X$. Thus f is onto and $\mathfrak{A} \cong \mathfrak{C} \mathrm{m} \mathfrak{A} \mathrm{t} \mathfrak{A}$.

(3) We show $\mathfrak{A} \mathrm{t} \mathfrak{A}$ satisfies 2.2(a)–(c). The proof of 2.2(a), (b) and half of (c) is the same as the first part of the proof of 3.8. For the other half of 2.2(c) we must prove, given $x \in \mathrm{At} \, \mathfrak{A}$, that $\langle x, w, x \rangle \in C^{(\mathfrak{A})}$ for some $w \in I^{(\mathfrak{A})}$. This follows from $x = x; 1' = x; \Sigma I^{(\mathfrak{A})} = \Sigma \{x; w : w \in I^{(\mathfrak{A})}\}$. Thus $\mathfrak{C} \mathrm{m} \mathfrak{A} \mathrm{t} \mathfrak{A} \in NA$.

(4) We will prove that if $\mathfrak{A}$ is a *WA*, *SA*, or *RA*, then $\mathfrak{A} \mathrm{t} \mathfrak{A}$ satisfies 2.2(d), (e), or (f), respectively.

Let $\mathfrak{A} \in WA$, $w \in I^{(\mathfrak{A})}$, and $\langle w, x, x \rangle$, $\langle x, y, z \rangle \in C^{(\mathfrak{A})}$. Then $z \leqslant x; y \leqslant w; x; y \leqslant w; 1; 1 = w; 1$ by 1.13(3) and the WL, so $z \leqslant w; \Sigma \text{At } \mathfrak{A} = \Sigma\{w; u : u \in \text{At } \mathfrak{A}\}$. Hence $z \leqslant w; u$ for some $u \in \text{At } \mathfrak{A}$. However $z \leqslant w; u \leqslant 1'; u = u$, so $z = u$, and $\langle w, z, z \rangle \in C^{(\mathfrak{A})}$. Thus 2.2(d) holds in $\mathfrak{A} t \mathfrak{A}$ and $\mathfrak{Cm } \mathfrak{A} t \mathfrak{A} \in WA$.

Let $\mathfrak{A} \in SA$, and $\langle v, w, x \rangle$, $\langle x, y, z \rangle \in C^{(\mathfrak{A})}$. Then $z \leqslant x; y \leqslant v; w; y \leqslant v; 1; 1 = v; 1$ by 1.13(3) and the SL, so $z \leqslant v; \Sigma \text{At } \mathfrak{A} = \Sigma\{v; u : u \in \text{At } \mathfrak{A}\}$. Hence $z \leqslant v; u$ for some $u \in \text{At } \mathfrak{A}$, and so $\langle v, u, z \rangle \in C^{(\mathfrak{A})}$. Thus $\mathfrak{A} t \mathfrak{A}$ satisfies 2.2(e) and $\mathfrak{Cm } \mathfrak{A} t \mathfrak{A} \in SA$.

Let $\mathfrak{A} \in RA$, and $\langle v, w, x \rangle$, $\langle x, y, z \rangle \in C^{(\mathfrak{A})}$. Then $z \leqslant z; y \leqslant v; w; y = v; (w; y)$ by 1.13(3) and the AL, so $z \leqslant v; \Sigma\{u; w; y \geqslant u \in \text{At } \mathfrak{A}\}$. Hence, for some $u \in \text{At } \mathfrak{A}$, $u \leqslant w; y$ and $z \leqslant v; u$, so $\langle w, y, u \rangle$, $\langle v, u, z \rangle \in C^{(\mathfrak{A})}$. Thus $\mathfrak{A} t \mathfrak{A}$ satisfies 2.2(f), and $\mathfrak{Cm } \mathfrak{A} t \mathfrak{A} \in RA$.

Since $\mathfrak{A}$ is isomorphic to a subalgebra of $\mathfrak{Cm } \mathfrak{A} t \mathfrak{A}$, it follows from the arguments above that $\mathfrak{Cm } \mathfrak{A} t \mathfrak{A}$ is a WA, SA, or RA iff $\mathfrak{A}$ is a WA, SA, or RA, respectively. Certainly $\mathfrak{A} \in RRA$ if $\mathfrak{Cm } \mathfrak{A} t \mathfrak{A} \in RRA$, so we need only prove $\mathfrak{Cm } \mathfrak{A} t \mathfrak{A} \in RRA$ whenever $\mathfrak{A} \in RRA$.

Suppose f is an embedding of $\mathfrak{A}$ into $\mathfrak{Sb } E$ for some equivalence relation E. Let $FX = \bigcup_{x \in X} fx$ for every $X \subseteq \text{At } \mathfrak{A}$. Then it is easy to check that F embeds $\mathfrak{Cm } \mathfrak{A} t \mathfrak{A}$ into $\mathfrak{Sb } E$, so $\mathfrak{Cm } \mathfrak{A} t \mathfrak{A} \in RRA$.

Theorem 3.13 shows that every complete and atomic NA is isomorphic to a complex algebra. In particular, every finite NA is isomorphic to the complex algebra of its atom structure. Obviously not every NA has this property, since there are incomplete NA's. However every atomic (and possibly incomplete) NA is isomorphic to a subalgebra of a complex algebra, and we shall see later (in 4.3) that this conclusion can be extended to all NA's.

4. Perfect extensions. The notion of perfect extension is defined for BA's with operators in Definition 2.14 of [**6**], and applies to NA's by 1.12.

THEOREM 4.1. *Let $\mathfrak{A}$, $\mathfrak{B} \in NA$. Then $\mathfrak{A}$ is a perfect extension of $\mathfrak{B}$ iff the following conditions are satisfied*:
(1) $\mathfrak{B} \subseteq \mathfrak{A}$,
(2) *$\mathfrak{A}$ is complete and atomic,*
(3) *if $X \subseteq B$ and $\Sigma^{(\mathfrak{A})}X = 1$, then $\Sigma^{(\mathfrak{A})}Y = 1$ for some finite $Y \subseteq X$,*
(4) *if $a, b \in \text{At } \mathfrak{A}$ then $a; b = \Pi^{(\mathfrak{A})}\{x; y : a \leqslant x \in B, b \leqslant y \in B\}$.*

PROOF. It follows from Definition 2.14 of [**6**] that $\mathfrak{A}$ is a perfect extension of $\mathfrak{B}$ iff (1)–(4) hold and
(5) if $a, b \in \text{At } \mathfrak{A}$ and $a \neq b$, then there is some $x \in B$ such that $a \leqslant x$ and $b \cdot x = 0$,
(6) if $a \in \text{At } \mathfrak{A}$, then $a^{\cup} = \Pi^{(\mathfrak{A})}\{x^{\cup} : a \leqslant x \in B\}$.
We therefore need only show that (5) and (6) are consequences of (1)–(4).

Let $a \in \text{At } \mathfrak{A}$. We will use (4) to show $a = \Pi^{(\mathfrak{A})}\{z : a \leqslant z \in B\}$. By 3.12, $\mathfrak{Cm } \mathfrak{A} t \mathfrak{A} \cong \mathfrak{A} \in NA$, so $\mathfrak{A} t \mathfrak{A}$ satisfies 2.2(c). Hence $\langle a, e, a \rangle \in C^{(\mathfrak{A})}$ for some $e \in I^{(\mathfrak{A})}$. Then

$a \leqslant a; e \leqslant a; 1' = a$, so $a = a; e = \Pi^{(\mathfrak{A})}\{x; y : a \leqslant x \in B,\ e \leqslant y \in B\}$ by (4). However, $\{x; y : a \leqslant x \in B,\ e \leqslant y \in B\} \subseteq \{z : a \leqslant z \in B\}$, for if $a \leqslant x \in B$ and $e \leqslant y \in B$, then $a = a; e \leqslant x; y \in B$. Consequently

$$a \leqslant \overset{(\mathfrak{A})}{\prod}\ \{z : a \leqslant z \in B\} \leqslant \overset{(\mathfrak{A})}{\prod}\ \{x; y : a \leqslant x \in B,\ e \leqslant y \in B\} = a,$$

so $a = \Pi^{(\mathfrak{A})}\{z : a \leqslant z \in B\}$. Using this equation and 1.11 we have

$$a^{\cup} = \left(\overset{(\mathfrak{A})}{\prod}\ \{a \leqslant z \in B\}\right)^{\cup} = \overset{(\mathfrak{A})}{\prod}\ \{z^{\cup} : a \leqslant z \in B\},$$

so (6) holds. Suppose $b \in \text{At } \mathfrak{A}$ and $b \neq a$. Then $0 = b \cdot a = \Pi^{(\mathfrak{A})}\{b \cdot z : a \leqslant z \in B\}$, so $b \cdot z = 0$ for some $z \in B$ such that $a \leqslant z$, i.e. (5) holds.

THEOREM 4.2. *Let* $\mathfrak{B} \in NA$. *Then there is some* $\mathfrak{A} \in NA$ *such that*

(1) $\mathfrak{A}$ *is a perfect extension of* $\mathfrak{B}$,

(2) *if* $\mathfrak{A}'$ *is a perfect extension of* $\mathfrak{B}$, *then there is a unique isomorphism h of* $\mathfrak{A}'$ *onto* $\mathfrak{A}$ *such that* $\text{Id}_B = B \restriction h$,

(3) $\mathfrak{A}$ *is a WA, SA, or RA iff* $\mathfrak{B}$ *is a WA, SA, or RA, respectively.*

PROOF. Theorem 2.15 of [6] insures that there is a *BA* with operators $\mathfrak{A}$ which is a perfect extension of $\mathfrak{B}$, and 2.17 of [6] insures that $\mathfrak{A}$ is unique in the sense of (2). We wish to use 2.18 of [6] to show that $\mathfrak{A} \in NA$ and (3) holds. Since complementation is not taken as a fundamental operation of *BA*'s in [6], the term "equationally definable class" in 2.18 refers to a class of *BA*'s with operators which is definable by equations containing no operation symbol for complementation. It follows from 1.4 that *NA*, *WA*, *SA*, and *RA* are such classes, so 2.18 applies.

It is not known whether *RRA* can be defined by equations without using complementation. Nevertheless, D. Monk proved that perfect extensions of representable *RA*'s are representable (as reported in [12, p. 66]).

THEOREM 4.3. *Let* $\mathfrak{A} \in NA$. *Then there is a structure* $\mathfrak{U} = \langle U, C, f, I \rangle$, *where* $C \subseteq {}^{3}U$, $f \in {}^{U}U$, *and* $I \subseteq U$, *such that*

(1) $\mathfrak{U}$ *satisfies 2.2(a)–(c), i.e.* $\mathfrak{Cm}\ \mathfrak{U} \in NA$,

(2) $\mathfrak{A}$ *is isomorphic to a subalgebra of* $\mathfrak{Cm}\ \mathfrak{U}$,

(3) $\mathfrak{A} \cong \mathfrak{Cm}\ \mathfrak{U}$ *iff* $\mathfrak{A}$ *is complete and atomic*,

(4) $\mathfrak{U}$ *satisfies 2.2(d), (e), or (f) iff* $\mathfrak{A}$ *is a WA, SA, or RA, respectively.*

PROOF. If $\mathfrak{A}$ is atomic, let $\mathfrak{U} = \mathfrak{A} t\ \mathfrak{A}$ and apply 3.13 and 2.2. Otherwise, let $\mathfrak{U} = \mathfrak{A} t\ \mathfrak{B}$ where $\mathfrak{B}$ is a perfect extension of $\mathfrak{A}$. Then $\mathfrak{B} \cong \mathfrak{Cm}\ \mathfrak{U}$ by 3.13, and (1)–(3) follow from 2.2 and 4.2.

Theorem 4.3 may be regarded as a representation theorem, even though there are nonrepresentable *NA*'s. The analogous theorem for cylindric algebras is 2.7.43 of [4], and the comments that follow, in Remarks 2.7.46, also apply to 4.3, with certain obvious changes.

5. Relativization. In this section we consider a generalization of the notion of relativization from *BA*'s to *NA*'s. A similar (but different) construction is introduced

in Definition 4.8 of [7]. The analogous construction for cylindric algebras is studied in §2.2 of [4].

DEFINITION 5.1. Let $\mathfrak{A} \in NA$ and $x \in A$. Then $\mathrm{Rl}_x \, \mathfrak{A} = \{y : x \geqslant y \in A\}$ and $\mathfrak{Rl}_x \, \mathfrak{A} = \langle \mathrm{Rl}_x \, \mathfrak{A}, +, \cdot, \,_x-, 0, x, ;^x, \cup^x, 1_x'\rangle$, where $_x{-}y = -y \cdot x$, $y;^x z = y; z \cdot x$, and $y^{\cup x} = y^\cup \cdot x$ for all $y, z \in \mathrm{Rl}_x \, \mathfrak{A}$. $\mathfrak{Rl}_x \, \mathfrak{A}$ is called the *relativization of $\mathfrak{A}$ to x.*

Some elementary consequences of this definition are given in the next two theorems.

THEOREM 5.2. *Let $\mathfrak{A}, \mathfrak{B} \in NA$ and $x, y \in A$.*
(1) $\mathfrak{Rl}_1 \, \mathfrak{A} = \mathfrak{A}$.
(2) *If $x \leqslant y$ then $\mathfrak{Rl}_x \, \mathfrak{Rl}_y \, \mathfrak{A} = \mathfrak{Rl}_x \, \mathfrak{A}$.*
(3) *If $\mathfrak{A} \subseteq \mathfrak{B}$ then $\mathfrak{Rl}_x \, \mathfrak{A} \subseteq \mathfrak{Rl}_x \, \mathfrak{B}$.*

THEOREM 5.3. *Let E be an equivalence relation.*
(1) *If $E \subseteq U \times U$ then $\mathfrak{Sb} \, E = \mathfrak{Rl}_E \, \mathfrak{Re} \, U$.*
(2) *If $U \in (\mathrm{Fd} \, E)/E$ then $\mathfrak{Re} \, U = \mathfrak{Rl}_{U \times U} \, \mathfrak{Sb} \, E$.*

Suppose $\mathfrak{A} \in NA$ and $x \in A$. What conditions on x are equivalent to $\mathfrak{Rl}_x \, \mathfrak{A} \in NA$? The answer is given in 5.5 and motivates the following definition.

DEFINITION 5.4. Let $\mathfrak{A} \in NA$. Then $\mathrm{Sr} \, \mathfrak{A} = \{x : x \in A$ and $x^\cup = x = (1' \cdot x); x\}$. We call x a *symmetric-reflexive element of $\mathfrak{A}$* iff $x \in \mathrm{Sr} \, \mathfrak{A}$.

Suppose E is an equivalence relation and $R \subseteq E$. Then $R \in \mathrm{Sr} \, \mathfrak{Sb} \, E$ iff R is symmetric ($R^{-1} = R$) and reflexive over its field ($\mathrm{Id}_{\mathrm{Fd} \, R} \subseteq R$).

THEOREM 5.5. *Let $\mathfrak{A} \in NA$ and $x \in A$. Then $\mathfrak{Rl}_x \, \mathfrak{A} \in NA$ iff $x \in \mathrm{Sr} \, \mathfrak{A}$.*

PROOF. Let $\mathfrak{Rl}_x \, \mathfrak{A} \in NA$. Then $x^{\cup x} = x$ by 1.13(9), so $x = x^\cup \cdot x$. Using 1.13(8), (11) we get $x^\cup = (x^\cup \cdot x)^\cup = x^{\cup\cup} \cdot x^\cup = x \cdot x^\cup = x$. Also $x = 1_x';^x x$ by the IL, so $x = (1' \cdot x); x \cdot x \leqslant (1' \cdot x); x \leqslant 1'; x = x$ by 1.13(3) and the IL. Thus $x^\cup = x = (1' \cdot x); x$, so $x \in \mathrm{Sr} \, \mathfrak{A}$.

Assume $x \in \mathrm{Sr} \, \mathfrak{A}$. Notice that $y^{\cup x} = y^\cup$ whenever $y \in \mathrm{Rl}_x \, \mathfrak{A}$, since $y^{\cup x} = y^\cup \cdot x = y^\cup \cdot x^\cup = (y \cdot x)^\cup = y^\cup$, and $w;^x y \cdot z = w; y \cdot z$ whenever $z \in \mathrm{Rl}_x \, \mathfrak{A}$. Consequently the Peircean laws for $\mathfrak{Rl}_x \, \mathfrak{A}$ follow immediately from the Peircean laws for $\mathfrak{A}$. For the IL in $\mathfrak{Rl}_x \, \mathfrak{A}$, we first note that $1_x';^x y = (1' \cdot x); y \cdot x \leqslant 1'; y = y$, for $y \in \mathrm{Rl}_x \, \mathfrak{A}$. Then we get

$$y = y \cdot x = y \cdot (1' \cdot x); x \cdot x$$
$$\leqslant (1' \cdot x); \left(x \cdot (1' \cdot x)^\cup ; y\right) \cdot x \qquad 1.4(1)$$
$$\leqslant (1' \cdot x); \left(1'^\cup ; y\right) \cdot x \qquad 1.13(3), (12)$$
$$= 1_x';^x y \qquad 1.13(14), \text{IL.}$$

Thus $y = 1_x';^x y$ for every $y \in \mathrm{Rl}_x \, \mathfrak{A}$. Now if $y \in \mathrm{Rl}_x \, \mathfrak{A}$ then $y^\cup \in \mathrm{Rl}_x \, \mathfrak{A}$ since $y^\cup \leqslant x^\cup = x$ by 1.13(12), so $y^\cup = 1_x';^x y^\cup$. Then

$$y = y^{\cup\cup} = \left(1_x';^x y^\cup\right)^\cup = \left((1' \cdot x); y^\cup \cdot x\right)^\cup$$
$$= y; (1' \cdot x^\cup) \cdot x^\cup = y; (1' \cdot x) \cdot x = y;^x 1_x'$$

by 1.13(8), (11), (13), (14). Thus the IL holds in $\mathfrak{Rl}_x \, \mathfrak{A}$, and $\mathfrak{Rl}_x \, \mathfrak{A} \in NA$.

DEFINITION 5.6. If $K \subseteq NA$, then $\mathrm{Rl}\, K = \{\mathfrak{Rl}_x\, \mathfrak{A} : \mathfrak{A} \in K \text{ and } x \in \mathrm{Sr}\, \mathfrak{A}\}$.

THEOREM 5.7. *Let* $K, L \subseteq NA$.

(1) *If* $K \subseteq L$ *then* $\mathrm{Rl}\, K \leqslant \mathrm{Rl}\, L$.

(2) $I\, \mathrm{Rl}\, K = \mathrm{Rl}\, IK$.

(3) $K \subseteq \mathrm{Rl}\, K$.

(4) $\mathrm{Rl}\, \mathrm{Rl}\, K \subseteq \mathrm{Rl}\, K$.

(5) $\mathrm{Rl}\, SK \subseteq S\, \mathrm{Rl}\, K$.

(6) Rl *and* $S\, \mathrm{Rl}$ *are closure operators on classes of* NA's.

PROOF. (1) is obvious, and (2) is clear by general algebra. If $\mathfrak{A} \in NA$ then $1 \in \mathrm{Sr}\, \mathfrak{A}$ by 1.13(9) and the IL, so (3) follows from 5.2(1). Let $\mathfrak{A} \in \mathrm{Rl}\, \mathrm{Rl}\, K$. Then $\mathfrak{A} = \mathfrak{Rl}_x\, \mathfrak{Rl}_y\, \mathfrak{B}$ for some $\mathfrak{B} \in K$, $y \in \mathrm{Sr}\, \mathfrak{B}$, and $x \in \mathrm{Sr}\, \mathfrak{Rl}_y\, \mathfrak{B}$. Since $\mathfrak{A} = \mathfrak{Rl}_x\, \mathfrak{B}$ by 5.2(2), we only need $x \in \mathrm{Sr}\, \mathfrak{B}$. From $x \in \mathrm{Sr}\, \mathfrak{Rl}_y\, \mathfrak{B}$ we get $x = x^{\cup y} = x^{\cup} \cdot y = x^{\cup} \cdot y^{\cup} = (x \cdot y)^{\cup} = x^{\cup}$ since $x \leqslant y$, and $x = (1'_y \cdot x) ;^y x = (1' \cdot y \cdot x) ; x \cdot y \leqslant (1' \cdot x) ; x \leqslant 1' ; x = x$, so $x \in \mathrm{Sr}\, \mathfrak{B}$. Thus (4) holds. (5) follows from 5.2(3) and the fact that $\mathrm{Sr}\, \mathfrak{A} \subseteq \mathrm{Sr}\, \mathfrak{B}$ whenever $\mathfrak{A} \subseteq \mathfrak{B}$. Finally, (6) follows from (1), (3)–(5).

THEOREM 5.8. (1) $\mathrm{Rl}\, NA = NA$.

(2) $\mathrm{Rl}\, WA = WA$.

PROOF. Part (1) follows from 5.5 and 5.7(3). We shall prove $\mathrm{Rl}\, WA \subseteq WA$, from which (2) follows by 5.7(3).

Let $\mathfrak{A} \in WA$ and $x \in \mathrm{Sr}\, \mathfrak{A}$. Since $\mathfrak{Rl}_x\, \mathfrak{A} \in NA$ by 5.5, it suffices to show that $\mathfrak{Rl}_x\, \mathfrak{A}$ satisfies the WL. Let $y \in \mathrm{Rl}_x\, \mathfrak{A}$. Then $(1'_x \cdot y) ;^x x \leqslant (1'_x \cdot y) ;^x x ;^x x$ by 1.13(4), and

$$
\begin{aligned}
(1'_x \cdot y) ;^x x ;^x x &= ((1' \cdot x \cdot y) ; x \cdot x) ; x \cdot x \\
&\leqslant (1' \cdot y) ; 1 ; 1 \cdot x && 1.13(3) \\
&= (1' \cdot y) ; 1 \cdot x && \text{WL} \\
&\leqslant (1' \cdot y) ; \left(1 \cdot (1' \cdot y)^{\cup} ; x\right) \cdot x && 1.4(1) \\
&\leqslant (1' \cdot y) ; x \cdot x && 1.13(3), (12), (14), \text{IL} \\
&= (1'_x \cdot y) ;^x x.
\end{aligned}
$$

The next theorem shows that SA, RA, and RRA are not closed under relativization.

THEOREM 5.9. *Let* U *be a set, and* $R \in \mathrm{Sr}\, \mathfrak{Re}\, U$.

(1) R *is transitive iff* $\mathfrak{Rl}_R\, \mathfrak{Re}\, U = \mathfrak{Sb}\, R$.

(2) R *is not transitive iff* $\mathfrak{Rl}_R\, \mathfrak{Re}\, U \in WA \sim SA$.

PROOF. R is symmetric and reflexive over its field, since $R \in \mathrm{Sr}\, \mathfrak{Re}\, U$. So if R is transitive, then R is an equivalence relation and $\mathfrak{Rl}_R\, \mathfrak{Re}\, U = \mathfrak{Sb}\, R$ by 5.3(1).

Suppose R is not transitive. We know that $\mathfrak{Rl}_R\, \mathfrak{Re}\, U \in WA$ by 5.8(2). Obviously $\mathfrak{Rl}_R\, \mathfrak{Re}\, U$ is complete and atomic, so $\mathfrak{Rl}_R\, \mathfrak{Re}\, U \cong \mathfrak{Cm}\, \mathfrak{At}\, \mathfrak{Rl}_R\, \mathfrak{Re}\, U$ by 3.13(2). We shall show that $\mathfrak{At}\, \mathfrak{Rl}_R\, \mathfrak{Re}\, U$ fails to satisfy 2.2(e), and hence $\mathfrak{Rl}_R\, \mathfrak{Re}\, U \notin SA$. Note that $\mathrm{At}\, \mathfrak{Rl}_R\, \mathfrak{Re}\, U = \{\{\langle a, b \rangle\} : \langle a, b \rangle \in R\}$. Since R is not transitive, there are a, b,

$c \in U$ such that $\langle a, b \rangle$, $\langle b, c \rangle \in R$ but $\langle a, c \rangle \notin R$. Set $v = \{\langle b, a \rangle\}$, $w = \{\langle a, b \rangle\}$, $x = \{\langle b, b \rangle\}$, and $y = \{\langle b, c \rangle\}$. Then w, $y \in \text{At } \Re\mathfrak{l}_R \Re e \, U$, and v, $x \in \text{At } \Re\mathfrak{l}_R \Re e \, U$ since R is symmetric and reflexive. Then $\langle v, w, x \rangle$, $\langle x, y, y \rangle \in C^{(\Re\mathfrak{l}_R \Re e \, U)}$ but there is no $u \in \text{At } \Re\mathfrak{l}_R \Re e \, U$ such that $\langle v, u, y \rangle \in C^{(\Re\mathfrak{l}_R \Re e \, U)}$.

We have seen that NA and WA are closed under Rl and SRl, while SA, RA, and RRA are not; in fact, Rl RRA and SRl RRA contain complete atomic WA's which are not SA's. The rest of this section is devoted to the problem of characterizing Rl RRA and SRl RRA. We will show that every complete atomic WA is in Rl RRA, and hence $WA = \text{SRl } RRA$. The proof is divided into two parts. After a definition and two lemmas we will show in 5.13 that every complete atomic WA is the relativization of a complete atomic WA which satisfies a special condition. Then, after another definition and four more lemmas, we will show in 5.19 that any atomic WA satisfying the special condition is in RRA.

DEFINITION 5.10. Let $\mathfrak{A} \in NA$. Then $x^d = x; x^{\cup} \cdot 1$' and $x^r = x^{\cup}; x \cdot 1$' for every $x \in A$.

LEMMA 5.11. *Let* $\mathfrak{A} \in NA$. *Then for every* $x \in A$:
(1) $(x^{\cup})^d = x^r$ *and* $(x^{\cup})^r = x^d$,
(2) *if* $x \leqslant 1$', *then* $x = x^d = x^r$,
(3) $x = x^d; x = x; x^r$.

PROOF. (1) follows immediately from 1.13(8). Suppose $x \leqslant 1$'. Then, using 1.13(3), (12), (14), 1.4(1), and the IL, we get $x^d = x; x^{\cup} \cdot 1$' $\leqslant x; 1$'$^{\cup} \cdot 1$' $\leqslant x = x; 1$' $\cdot 1$' $\leqslant x; (1$' $\cdot x^{\cup}; 1$') $\cdot 1$' $\leqslant x; x^{\cup} \cdot 1$' $= x^d$, so $x^d = x$. Now $x = x^{\cup}$ by 1.13(16), so $x^r = (x^{\cup})^d = x^d$ by part (1). Thus (2) holds. For (3), we use 1.4(2), 1.13(3), and the IL to get $x^d; x \leqslant 1$'; $x = x = 1$'; $x \cdot x \leqslant (1$' $\cdot x; x^{\cup}); x = x^d; x$. The proof of $x = x; x^r$ is similar.

LEMMA 5.12. *Let* $\mathfrak{A} \in WA$. *Then for all* $x, y, z \in \text{At } \mathfrak{A}$ *and* $w \in I^{(\mathfrak{A})}$,
(1) $x^d, x^r \in \text{At } \mathfrak{A}$,
(2) $w = x^d$ *iff* $x \leqslant w; x$ *iff* $\langle w, x, x \rangle \in C^{(\mathfrak{A})}$,
(3) $w = x^r$ *iff* $x \leqslant x; w$ *iff* $\langle x, w, x \rangle \in C^{(\mathfrak{A})}$,
(4) *if* $x \leqslant y; z$ *then* $x^d = y^d$, $y^r = z^d$, *and* $x^r = z^r$.

PROOF. Part (1) is a restatement of 3.5. If $w = x^d$, then $x \leqslant w; x$ by 5.11(3). Suppose $x \leqslant w; x$. Then $0 \neq x = w; x \cdot x \leqslant (w \cdot x; x^{\cup}); x$ by 1.4(2). Since $w = w \cdot 1$' and ; is normal, we get $0 \neq w \cdot x; x^{\cup} = w \cdot x^d$. Hence $w = x^d$ by part (1). We have shown $w = x^d$ iff $x \leqslant w; x$. This completes the proof of (2), since $x \leqslant w; x$ and $\langle w, x, x \rangle \in C^{(\mathfrak{A})}$ are equivalent by the definition of $C^{(\mathfrak{A})}$. The proof of (3) is similar.

Assume $x \leqslant y; z$. Then

$$
\begin{aligned}
x &= y; z \cdot x \\
&= y^d; y; z \cdot x && \text{5.11(3)} \\
&\leqslant y^d; 1; 1 \cdot x && \text{1.13(3)} \\
&= y^d; 1 \cdot x && \text{WL} \\
&\leqslant y^d; \left(1 \cdot (y^d)^{\cup}; x\right) && \text{1.4(1)} \\
&\leqslant y^d; x && \text{1.13(3), (12), (14), IL,}
\end{aligned}
$$

so $x^d = y^d$ by (1) and (2). From $0 \neq x = y; z \cdot x$ we get $y^{\cup} ; x \cdot z \neq 0$ by the LPL, and $0 \neq z^{\cup} ; y^{\cup} \cdot x^{\cup}$ by 1.13(7), (8), (11), (13). Hence $z \leqslant y^{\cup} ; x$ and $x^{\cup} \leqslant z^{\cup} ; y^{\cup}$ since $x^{\cup} \in \mathrm{At}\ \mathfrak{A}$ by 3.4. By the argument above and 5.11(1), we get $z^d = (y^{\cup})^d = y^r$ and $x^r = (x^{\cup})^d = (z^{\cup})^d = z^r$.

THEOREM 5.13. *Assume $\mathfrak{A} \in WA$ and $\mathfrak{A}$ is complete and atomic. Then $\mathfrak{A} \cong \mathfrak{Rl}_a \mathfrak{B}$ for some $\mathfrak{B} \in WA$ and some $a \in \mathrm{Sr}\ \mathfrak{B}$, where $\mathfrak{B}$ is complete, atomic, and satisfies the following condition:*

(∗) *for all $u, v \in I^{(\mathfrak{B})}$ there is some $x \in \mathrm{At}\ \mathfrak{B} \sim I^{(\mathfrak{B})}$ such that $x^d = u$, $x^r = v$, and $x \leqslant y; z$ whenever $y, z \in \mathrm{At}\ \mathfrak{B} \sim I^{(\mathfrak{B})}$, $y^d = u$, $y^r = z^d$, and $z^r = v$.*

PROOF. We shall assume, without loss of generality, that At $\mathfrak{A}$ and $I^{(\mathfrak{A})} \times I^{(\mathfrak{A})}$ are disjoint. Set $U = \mathrm{At}\ \mathfrak{A} \cup (I^{(\mathfrak{A})} \times I^{(\mathfrak{A})})$. Define $f \in {}^U U$ as follows:

$$fx = \begin{cases} x^{\cup} & \text{if } x \in \mathrm{At}\ \mathfrak{A}, \\ \langle v, u \rangle & \text{if } x = \langle u, v \rangle, u, v \in I^{(\mathfrak{A})}. \end{cases}$$

Let D be the subset of ${}^3 U$ consisting of all triples of the following types:

(Ia)	$\left\langle x, y, \left\langle x^d, y^r \right\rangle \right\rangle$	$x^r = y^d, x, y \in \mathrm{At}\ \mathfrak{A} \sim I^{(\mathfrak{A})},$
(Ib)	$\left\langle x, \left\langle x^r, y^r \right\rangle, y \right\rangle$	$x^d = y^d, x, y \in \mathrm{At}\ \mathfrak{A} \sim I^{(\mathfrak{A})},$
(Ic)	$\left\langle \left\langle y^d, x^d \right\rangle, x, y \right\rangle$	$x^r = y^r, x, y \in \mathrm{At}\ \mathfrak{A} \sim I^{(\mathfrak{A})},$
(IIa)	$\left\langle \left\langle x^d, u \right\rangle, \left\langle u, x^r \right\rangle, x \right\rangle$	$u \in I^{(\mathfrak{A})}, x \in \mathrm{At}\ \mathfrak{A},$
(IIb)	$\left\langle x, \left\langle x^r, u \right\rangle, \left\langle x^d, u \right\rangle \right\rangle$	$u \in I^{(\mathfrak{A})}, x \in \mathrm{At}\ \mathfrak{A},$
(IIc)	$\left\langle \left\langle u, x^d \right\rangle, x, \left\langle u, x^r \right\rangle \right\rangle$	$u \in I^{(\mathfrak{A})}, x \in \mathrm{At}\ \mathfrak{A},$
(III)	$\left\langle \left\langle u, v \right\rangle, \left\langle v, w \right\rangle, \left\langle u, w \right\rangle \right\rangle$	$u, v, w \in I^{(\mathfrak{A})}.$

Let $C' = C^{(\mathfrak{A})} \cup D$. Finally, let $\mathfrak{U} = \langle U, C', f, I^{(\mathfrak{A})} \rangle$ and $\mathfrak{B} = \mathfrak{Cm}\ \mathfrak{U}$. Notice that $C' \cap {}^3 \mathrm{At}\ \mathfrak{A} = C^{(\mathfrak{A})}$ and the restriction of f to At $\mathfrak{A}$ is ${}^{\cup}$. Consequently $\mathfrak{A} \mathfrak{t}\ \mathfrak{A}$ is a substructure of $\mathfrak{U}$.

We shall prove that $\mathfrak{Cm}\ \mathfrak{U} \in WA$ by verifying 2.2(a)–(d). For 2.2(a), (b), suppose $\langle x, y, z \rangle \in C'$. If $\langle x, y, z \rangle \in C^{(\mathfrak{A})}$, then $\langle fx, z, y \rangle = \langle x^{\cup}, z, y \rangle \in C^{(\mathfrak{A})} \subseteq C'$ and $\langle z, fy, x \rangle = \langle z, y^{\cup}, x \rangle \in C^{(\mathfrak{A})} \subseteq C'$ since $\mathfrak{A} \mathfrak{t}\ \mathfrak{A}$ satisfies 2.2(a), (b). If $\langle x, y, z \rangle$ is a triple of type (Ia), then $\langle x, y, z \rangle = \langle x, y, \langle x^d, y^r \rangle \rangle$ where $x^r = y^d$ and $x, y \in \mathrm{At}\ \mathfrak{A} \sim I^{(\mathfrak{A})}$. But then $(x^{\cup})^d = y^d$ by 5.11(1) and $x^{\cup}, y \in \mathrm{At}\ \mathfrak{A} \sim I^{(\mathfrak{A})}$ by 3.4 and 1.13(16), so $\langle fx, \langle x^d, y^r \rangle, y \rangle = \langle x^{\cup}, \langle (x^{\cup})^r, y^r \rangle, y \rangle$ is a triple of type (Ib). Similar observations show that every transform of a triple of type (I) is also type (I). It is even easier to see that if $\langle x, y, z \rangle$ is type (II) or (III), then so are all of its transforms (nothing more than the definition of f is needed). Thus $\mathfrak{U}$ satisfies 2.2(a), (b).

Next we verify 2.2(c) in one direction. Let $x \in U$. If $x \in \mathrm{At}\ \mathfrak{A}$, then $\langle x, w, x \rangle \in C^{(\mathfrak{A})} \subseteq C'$ for some $w \in I^{(\mathfrak{A})}$ since $\mathfrak{A} \mathfrak{t}\ \mathfrak{A}$ satisfies 2.2(c). Suppose $x = \langle u, v \rangle$ where $u, v \in I^{(\mathfrak{A})}$. Then $v = v^d = v^r$ by 5.11(2), so $\langle x, v, x \rangle = \langle \langle u, v \rangle, v, \langle u, v \rangle \rangle$ is a triple of type (IIc). Now, for the other direction, suppose $\langle x, w, y \rangle \in C'$ with $w \in I^{(\mathfrak{A})}$. If $x, y \in \mathrm{At}\ \mathfrak{A}$, then $\langle x, w, y \rangle \in C^{(\mathfrak{A})}$ and $x = y$ since $\mathfrak{A} \mathfrak{t}\ \mathfrak{A}$ satisfies 2.2(c). Suppose $x = \langle u, v \rangle$ where $u, v \in I^{(\mathfrak{A})}$. Then $\langle x, w, y \rangle$ must be type (IIc).

(Type (Ic) is ruled out since $w \notin \text{At } \mathfrak{A} \sim I^{(\mathfrak{A})}$.) Consequently $x = \langle u, w^d \rangle$ and $y = \langle u, w^r \rangle$. But $w = w^d = w^r$ by 5.11(2), so $x = y$. This completes the proof of 2.2(c) for $\mathfrak{U}$.

For 2.2(d) we must show $\langle w, z, z \rangle \in C'$ whenever $w \in I^{(\mathfrak{A})}$ and $\langle w, x, x \rangle$, $\langle x, y, z \rangle \in C'$. This certainly holds when $x, y, z \in \text{At } \mathfrak{A}$ since $\mathfrak{A}\text{t } \mathfrak{A}$ satisfies 2.2(d). There are seven other cases depending on the type of $\langle x, y, z \rangle$. These cases are listed below. Throughout the list we assume $q, u, v, w \in I^{(\mathfrak{A})}$ and $x, y \in \text{At } \mathfrak{A}$.

$$(1) \quad \langle w, x, x \rangle, \langle x, y, \langle x^d, y^r \rangle \rangle \in C', \qquad x^r = y^d,$$

$$(2) \quad \langle w, x, x \rangle, \langle x, \langle x^r, y^r \rangle, y \rangle \in C', \qquad x^d = y^d,$$

$$(3) \quad \langle w, \langle y^d, x^d \rangle, \langle y^d, x^d \rangle \rangle, \langle \langle y^d, x^d \rangle, x, y \rangle \in C', \qquad x^r = y^r,$$

$$(4) \quad \langle w, \langle x^d, u \rangle, \langle x^d, u \rangle \rangle, \langle \langle x^d, u \rangle, \langle u, x^r \rangle, x \rangle \in C',$$

$$(5) \quad \langle w, x, x \rangle, \langle x, \langle x^r, u \rangle, \langle x^d, u \rangle \rangle \in C',$$

$$(6) \quad \langle w, \langle u, x^d \rangle, \langle u, x^d \rangle \rangle, \langle \langle u, x^d \rangle, x, \langle u, x^r \rangle \rangle \in C',$$

$$(7) \quad \langle w, \langle u, v \rangle, \langle u, v \rangle \rangle, \langle \langle u, v \rangle, \langle v, q \rangle, \langle u, q \rangle \rangle \in C'.$$

Case (1). We get $w = x^d$ by 5.12(2), so $\langle w, \langle x^d, y^r \rangle, \langle x^d, y^r \rangle \rangle$ is a triple of type (IIb).

Case (2). By 5.12(2), $w = x^d$. But $x^d = y^d$, so $\langle w, y, y \rangle \in C^{(\mathfrak{A})}$ by 5.12(2).

Case (3). The first triple can only be type (IIb), so $y^d = w^d$. Then $y^d = w$ by 5.11(2), and $\langle w, y, y \rangle \in C^{(\mathfrak{A})}$ by 5.12(2).

Case (4). Similar to case (3).

Case (5). Similar to case (1).

Case (6). The first triple must be type (IIb), so $u = w^d = w$ by 5.11(2). Then $\langle w, \langle u, x^r \rangle, \langle u, x^r \rangle \rangle$ is type (IIb) as well.

Case (7). Similar to case (6).

This completes the proof of $\mathfrak{Cm } \mathfrak{U} \in WA$.

Consider $\text{At } \mathfrak{A}$ as an element of $\mathfrak{Cm } \mathfrak{U}$. We have $(\text{At } \mathfrak{U})^{\cup} = \{ fx : x \in \text{At } \mathfrak{A} \} = \{ x^{\cup} : x \in \text{At } \mathfrak{A} \} = \text{At } \mathfrak{A}$ by 3.4 and 1.13(8), and $(I^{(\mathfrak{A})} \cap \text{At } \mathfrak{A}); \text{At } \mathfrak{A} = I^{(\mathfrak{A})}; \text{At } \mathfrak{A} = \text{At } \mathfrak{A}$ by the IL for $\mathfrak{Cm } \mathfrak{U}$, so $\text{At } \mathfrak{A} \in \text{Sr } \mathfrak{Cm } \mathfrak{U}$. It is easy to check that $\mathfrak{R} I_{\text{At } \mathfrak{A}} \mathfrak{Cm } \mathfrak{U} = \mathfrak{Cm } \mathfrak{A}\text{t } \mathfrak{A}$. The crucial fact, which is needed to show that these two algebras have the same relative product, is that $C' \cap^3 \text{At } \mathfrak{A} = C^{(\mathfrak{A})}$. We have $\mathfrak{A} \cong \mathfrak{Cm } \mathfrak{A}\text{t } \mathfrak{A}$ by 3.13, so, setting $a = \text{At } \mathfrak{A}$ and $\mathfrak{B} = \mathfrak{Cm } \mathfrak{U}$, we get $\mathfrak{B} \in WA$, $a \in \text{Sr } \mathfrak{B}$, $\mathfrak{A} \cong \mathfrak{Cm } \mathfrak{A}\text{t } \mathfrak{A} = \mathfrak{R} I_{\text{At } \mathfrak{A}} \mathfrak{Cm } \mathfrak{U} = \mathfrak{R} I_a \mathfrak{B}$, and $\mathfrak{B}$ is complete and atomic. It remains to show that $\mathfrak{B}$ satisfies (*).

First notice that $\text{At } \mathfrak{B} = \{ \{ x \} : x \in \text{At } \mathfrak{A} \} \cup \{ \{ \langle u, v \rangle \} : u, v \in I^{(\mathfrak{A})} \}$. If $x \in \text{At } \mathfrak{A}$ then $\{ x \}^d = I^{(\mathfrak{A})} \cap \{ x \}; \{ x \}^{\cup} = \{ u : u \in I^{(\mathfrak{A})}$ and $\langle x, x^{\cup}, u \rangle \in C' \} = \{ u : u \in I^{(\mathfrak{A})}$ and $\langle u, x, x \rangle \in C^{(\mathfrak{A})} \}$ since $C' \cap^3 \text{At } \mathfrak{A} = C^{(\mathfrak{A})}$. Consequently $\{ x \}^d = \{ x^d \}$ by 5.12(2). Similarly, $\{ x \}^r = \{ x^r \}$. If $u, v \in I^{(\mathfrak{A})}$, then $\{ \langle u, v \rangle \}^d = I^{(\mathfrak{A})} \cap \{ \langle u, v \rangle \}$; $\{ \langle u, v \rangle \}^{\cup} = \{ w : w \in I^{(\mathfrak{A})}$ and $\langle w, \langle u, v \rangle, \langle u, v \rangle \rangle \in C' \}$. So $w \in \{ \langle u, v \rangle \}^d$ iff $w \in I^{(\mathfrak{A})}$ and $\langle w, \langle u, v \rangle, \langle u, v \rangle \rangle$ is a triple of type (IIb), which occurs iff $u = w^d = w$ by 5.11(2). Thus $\{ \langle u, v \rangle \}^d = \{ u \}$, and similarly $\{ \langle u, v \rangle \}^r = \{ v \}$.

Let u', $v' \in I^{(\mathfrak{B})}$. Then $u' = \{u\}$ and $v' = \{v\}$ for some $u, v \in I^{(\mathfrak{A})}$. Set $x' = \{\langle u, v \rangle\}$. Note that $x' \in \text{At } \mathfrak{B} \sim I^{(\mathfrak{B})}$. Let y', $z' \in \text{At } \mathfrak{B} \sim I^{(\mathfrak{B})}$, and assume

$$(**) \qquad (y')^d = \{u\}, \quad (y')^r = (z')^d, \quad \text{and} \quad (z')^r = \{v\}.$$

We must show $x' = \{\langle u, v \rangle\} \subseteq y'; z'$. The proof splits into four cases.

Case (a). $y' = \{y\}$, $z' = \{z\}$, y, $z \in \text{At } \mathfrak{A} \sim I^{(\mathfrak{A})}$. From $(**)$ we get $y^d = u$, $y^r = z^d$, and $z^r = v$. Then $\langle y, z, \langle u, v \rangle \rangle = \langle y, z, \langle y^d, z^r \rangle \rangle$ is a triple of type (Ia), so $x' \subseteq y'; z'$.

Case (b). $y' = \{y\}$, $y \in \text{At } \mathfrak{A} \sim I^{(\mathfrak{A})}$, $z' = \{\langle q, w \rangle\}$, q, $w \in I^{(\mathfrak{A})}$. Using $(**)$ we get $y^d = u$, $y^r = q$, and $w = v$. Then $\langle y, \langle q, w \rangle, \langle u, v \rangle \rangle = \langle y, \langle y^r, v \rangle, \langle y^d, v \rangle \rangle$ is a triple of type (IIb) and $x' \subseteq y'; z'$.

Case (c). $y' = \{\langle q, w \rangle\}$, q, $w \in I^{(\mathfrak{A})}$, $z' = \{z\}$, $z \in \text{At } \mathfrak{A} \sim I^{(\mathfrak{A})}$. From $(**)$ we have $q = u$, $w = z^d$, and $z^r = v$. Then $\langle \langle q, w \rangle, z, \langle u, v \rangle \rangle = \langle \langle u, z^d \rangle, z, \langle u, z^r \rangle \rangle$ is a triple of type (IIc), so $x' \subseteq y'; z'$.

Case (d). $y' = \{\langle q, w \rangle\}$, $z' = \{\langle s, t \rangle\}$, $q, w, s, t \in I^{(\mathfrak{A})}$. By $(**)$, $q = u$, $w = s$, and $t = v$, so $\langle \langle q, w \rangle, \langle s, t \rangle, \langle u, v \rangle \rangle = \langle \langle u, w \rangle, \langle w, v \rangle, \langle u, v \rangle \rangle$ is a triple of type (III), and hence $x' \subseteq y'; z'$.

DEFINITION 5.14. Assume $\mathfrak{A} \in NA$ and α is an ordinal. Then $L_\alpha \mathfrak{A}$ is the set of all $l \in {}^{\alpha \times \alpha}\text{At } \mathfrak{A}$ satisfying the following conditions for all $\kappa, \lambda, \mu < \alpha$.

(1) $l_{\kappa\lambda} \leq 1'$ iff $\kappa = \lambda$,
(2) $l_{\kappa\lambda}^{\cup} = l_{\lambda\kappa}$,
(3) $l_{\kappa\lambda} \leq l_{\kappa\mu}; l_{\mu\lambda}$.

The elements of $L_\alpha \mathfrak{A}$ are called $\mathfrak{A}$-*labellings* of α. Let $l \in L_\alpha \mathfrak{A}$. A *flaw* in l is a quadruple $\langle \kappa, \lambda, x, y \rangle$ where $\kappa, \lambda < \alpha$, $x, y \in \text{At } \mathfrak{A}$, $l_{\kappa\lambda} \leq x; y$, and for every $\mu < \alpha$, either $l_{\kappa\mu} \neq x$ or $l_{\mu\lambda} \neq y$. The labelling l is *complete* if there are no flaws in l.

LEMMA 5.15. *Assume* $\mathfrak{A} \in WA$, $\mathfrak{A}$ *is atomic,* l *is a complete* $\mathfrak{A}$-*labelling of* α, *and* l *maps* $\alpha \times \alpha$ *onto* $\text{At } \mathfrak{A}$. *Then* $\mathfrak{A}$ *is isomorphic to a subalgebra of* $\mathfrak{Re } \alpha$.

PROOF. Define $R(x) = \{\langle \kappa, \lambda \rangle : \kappa, \lambda < \alpha \text{ and } l_{\kappa\lambda} \leq x\}$ for all $x \in A$. Clearly, if $x \leq y$ then $R(x) \subseteq R(y)$. Suppose $x \nleq y$. Then $x \cdot -y \neq 0$, so $a \leq x \cdot -y$ for some $a \in \text{At } \mathfrak{A}$ since $\mathfrak{A}$ is atomic. Since l is onto, there are $\kappa, \lambda < \alpha$ such that $l_{\kappa\lambda} = a \leq x \cdot -y$. But then $\langle \kappa, \lambda \rangle \in R(x)$ and $\langle \kappa, \lambda \rangle \notin R(y)$, so $R(x) \nsubseteq R(y)$. We have shown

$$(1) \qquad x \leq y \quad \text{iff} \quad R(x) \subseteq R(y), \qquad \text{for all } x, y \in A.$$

Clearly $R(0) = \varnothing$ and $R(1) = \alpha \times \alpha$. Consequently R is an embedding of $\mathfrak{Bl } \mathfrak{A}$ into $\mathfrak{Bl } \mathfrak{Re } \alpha$.

We get $\text{Id}_\alpha = R(1')$ from 5.14(1), and $R(x^{\cup}) = R(x)^{-1}$ from 5.14(2). It follows from 5.14(3) that $R(x) \mid R(y) \subseteq R(x; y)$, while $R(x; y) \subseteq R(x) \mid R(y)$ since l is complete. Thus R is an embedding of $\mathfrak{A}$ into $\mathfrak{Re } \alpha$.

LEMMA 5.16. *If* $\mathfrak{A} \in WA$, $\mathfrak{A}$ *satisfies* 5.13(*), $l \in L_\alpha \mathfrak{A}$, *and* $\langle \kappa, \lambda, x, y \rangle$ *is a flaw in* l, *then there is some* $m \in L_{\alpha+1} \mathfrak{A}$ *such that* $l \subseteq m$, $m_{\kappa\alpha} = x$, *and* $m_{\alpha\lambda} = y$ (*i.e.* $\langle \kappa, \lambda, x, y \rangle$ *is not a flaw in* m).

PROOF. Since $\mathfrak{A}$ satisfies 5.13(*), there is a function a, mapping $I^{(\mathfrak{A})} \times I^{(\mathfrak{A})}$ into At $\mathfrak{A} \sim I^{(\mathfrak{A})}$, such that for all $u, v \in I^{(\mathfrak{A})}$ we have

(1) $a(u, v)^d = u$,

(2) $a(u, v)^r = v$,

(3) if $w, z \in$ At $\mathfrak{A}$, $w^d = u$, $w^r = z^d$, and $z^r = v$, then $a(u, v) \leqslant w; z$.

Using 3.4 and 5.12(1), we define $m \in {}^{(\alpha+1)\times(\alpha+1)}$At $\mathfrak{A}$ as follows:

$$m_{\mu\nu} = l_{\mu\nu} \quad \text{for all } \mu, \nu < \alpha,$$

$$m_{\kappa\alpha} = x, \qquad m_{\alpha\kappa} = x^\cup, \qquad m_{\alpha\lambda} = y,$$

$$m_{\lambda\alpha} = y^\cup, \qquad m_{\alpha\alpha} = x^r, \qquad m_{\mu\alpha} = a\big(l_{\mu\mu}, x^r\big),$$

$$m_{\alpha\mu} = a\big(l_{\mu\mu}, x^r\big)^\cup \quad \text{if } \mu < \alpha \text{ and } \mu \neq \kappa, \lambda.$$

We have apparently assumed $\kappa \neq \lambda$ in the definition of m. However, suppose $\kappa = \lambda$. Then $0 \neq l_{\kappa\kappa} = l_{\kappa\lambda} \leqslant x; y \cdot 1$' by 5.14(1), so $0 \neq x^\cup; 1' \cdot y = x^\cup \cdot y$ by the LPL, and $0 \neq x \cdot y^\cup$ by the RPL. Thus $y = x^\cup$ and $y^\cup = x$, so the definition of m is merely redundant.

The proof will be complete as soon as we show $m \in L_{\alpha+1} \mathfrak{A}$. To verify 5.14(2) for m it suffices, since $m \supseteq l \in L_\alpha \mathfrak{A}$, to show $l_{\alpha\alpha}^\cup = l_{\alpha\alpha}$, $l_{\mu\alpha}^\cup = l_{\alpha\mu}$, and $l_{\alpha\mu}^\cup = l_{\mu\alpha}$ for all $\mu < \alpha$. These formulas follow from 1.13(8), (16). Obviously m satisfies half of 5.14(1), namely $m_{\mu\mu} \leqslant 1$' for all $\mu < \alpha + 1$. We will prove the other half of 5.14(1) later.

Before continuing we make some observations which will simplify the proof that l satisfies 5.14(3).

(4) $$x^d = m_{\kappa\kappa}, \qquad x^r = y^d, \quad \text{and } y^r = m_{\lambda\lambda}.$$

PROOF. Since $\langle \kappa, \lambda, x, y \rangle$ is a flaw in l, we have $l_{\kappa\lambda} \leqslant x; y$, and so, by 5.12(4), $l_{\kappa\lambda}^d = x^d$, $x^r = y^d$, and $l_{\kappa\lambda}^r = y^r$. Since $l \in L_\alpha \mathfrak{A}$, we have $l_{\kappa\kappa}$, $l_{\lambda\lambda} \leqslant 1$', $l_{\kappa\lambda} \leqslant l_{\kappa\kappa}; l_{\kappa\lambda}$, and $l_{\kappa\lambda} \leqslant l_{\kappa\lambda}; l_{\lambda\lambda}$. Hence $l_{\kappa\lambda}^d = l_{\kappa\kappa} = m_{\kappa\kappa}$ and $l_{\kappa\lambda}^r = l_{\lambda\lambda} = m_{\lambda\lambda}$ by 5.12(2), (3).

Thus $x^d = l_{\kappa\lambda}^d = m_{\kappa\kappa}$ and $y^r = l_{\kappa\lambda}^r = m_{\lambda\lambda}$.

(5) $$m_{\mu\nu}^d = m_{\mu\mu} \quad \text{for all } \mu, \nu < \alpha + 1.$$

PROOF. Suppose $\mu, \nu < \alpha$. Then $m_{\mu\mu} = l_{\mu\mu} \leqslant 1$' and $m_{\mu\nu} = l_{\mu\nu} \leqslant l_{\mu\mu}; l_{\mu\nu} = m_{\mu\mu}; m_{\mu\nu}$, so $m_{\mu\nu}^d = m_{\mu\mu}$ by 5.12(2). Thus we may assume $\alpha \in \{\mu, \nu\}$. If $\alpha = \mu = \nu$, then $m_{\mu\nu}^d = m_{\alpha\alpha}^d = (x^r)^d = x^r = m_{\alpha\alpha} = m_{\mu\mu}$ by 5.11(2), so either $\mu < \alpha = \nu$ or $\nu < \alpha = \mu$.

Suppose $\mu < \alpha = \nu$. If $\mu \neq \kappa, \lambda$, then $m_{\mu\nu}^d = m_{\mu\alpha}^d = a(l_{\mu\mu}, x^r)^d = l_{\mu\mu} = m_{\mu\mu}$ by (1). If $\mu = \kappa$, then $m_{\mu\nu}^d = m_{\kappa\alpha}^d = x^d = m_{\kappa\kappa} = m_{\mu\mu}$ by (4). If $\mu = \lambda$, then $m_{\mu\nu}^d = m_{\lambda\alpha}^d = (y^\cup)^d = y^r = m_{\lambda\lambda} = m_{\mu\mu}$ by (4).

Suppose $\nu < \alpha = \mu$. If $\nu \neq \kappa, \lambda$, then $m_{\mu\nu}^d = m_{\alpha\nu}^d = (a(l_{\nu\nu}, x^r)^\cup)^d = a(l_{\nu\nu}, x^r)^r = x^r = m_{\alpha\alpha} = m_{\mu\mu}$ by (2). If $\nu = \kappa$ then $m_{\mu\nu}^d = m_{\alpha\kappa}^d = (x^\cup)^d = x^r = m_{\alpha\alpha} = m_{\mu\mu}$. If $\nu = \lambda$, then $m_{\mu\nu}^d = m_{\alpha\lambda}^d = y^d = x^r = m_{\alpha\alpha} = m_{\mu\mu}$ by (4).

(6) $$m_{\mu\nu}^r = m_{\nu\nu} \quad \text{for all } \mu, \nu < \alpha + 1.$$

PROOF. Using (5) and 5.14(2) for m, we have $m_{\mu\nu}^r = (m_{\mu\nu}^\cup)^d = m_{\nu\mu}^d = m_{\nu\nu}$.

(7) For all $\mu, \nu, \xi < \alpha + 1$, the six formulas which can be obtained from $m_{\mu\nu} \leqslant m_{\mu\xi}; m_{\xi\nu}$ by permuting the subscripts μ, ν, ξ are equivalent.

PROOF. This follows from the Peircean laws, $m \in {}^{(\alpha+1)\times(\alpha+1)}\text{At } \mathfrak{A}$, and 5.14(2) for m. For example, if $m_{\mu\nu} \leqslant m_{\mu\xi}; m_{\xi\nu}$, then $0 \neq m_{\mu\nu} = m_{\mu\xi}; m_{\xi\nu} \cdot m_{\mu\nu}$, so $0 \neq m_{\mu\xi}^{\cup}; m_{\mu\nu} \cdot m_{\xi\mu} = m_{\xi\mu}; m_{\mu\nu} \cdot m_{\xi\nu}$, and hence $m_{\xi\nu} \leqslant m_{\xi\mu}; m_{\mu\nu}$.

Now we can show m satisfies 5.14(3). Let μ, ν, $\xi < \alpha + 1$. We wish to show $m_{\mu\nu} \leqslant m_{\mu\xi}; m_{\xi\nu}$. Suppose μ, ν, ξ are not distinct. We may assume $\mu = \xi$ by (7). Then by (5) and 5.11(3), $m_{\mu\nu} = m_{\mu\nu}^d; m_{\mu\nu} = m_{\mu\mu}; m_{\mu\nu} = m_{\mu\xi}; m_{\xi\nu}$.

Suppose μ, ν, ξ are distinct. If μ, ν, $\xi < \alpha$, then $m_{\mu\nu} = l_{\mu\nu} \leqslant l_{\mu\xi}; l_{\xi\nu} = m_{\mu\xi}; m_{\xi\nu}$ since $l \in L_\alpha \mathfrak{A}$, so we may assume, by (7), that $\nu = \alpha$. Thus μ, $\xi < \alpha = \nu$. Suppose $\{\mu, \xi\} \neq \{\kappa, \lambda\}$. We may assume, by (7), that $\mu \neq \kappa$, λ. Then $m_{\mu\nu} = m_{\mu\alpha} = a(l_{\mu\mu}, x^r)$, and from (5) and (6) we get $l_{\mu\mu} = m_{\mu\mu} = m_{\mu\xi}^d$, $m_{\mu\xi}^r = m_{\xi\xi} = m_{\xi\nu}^d$, and $m_{\xi\alpha}^r = m_{\alpha\alpha} = x^r$, so by (1)–(3), $m_{\mu\nu} = a(l_{\mu\mu}, x^r) \leqslant m_{\mu\xi}; m_{\xi\nu}$. Suppose $\{\mu, \xi\} = \{\kappa, \lambda\}$. By (7) we may assume $\mu = \kappa$ and $\xi = \lambda$ since $\mu \neq \xi$. We have $l_{\kappa\lambda} \leqslant x; y$ since $\langle \kappa, \lambda, x, y \rangle$ is a flaw in l, so $m_{\mu\xi} = m_{\kappa\lambda} = l_{\kappa\lambda} \leqslant x; y = m_{\kappa\alpha}; m_{\alpha\lambda} = m_{\mu\nu}; m_{\nu\xi}$. Hence $m_{\mu\nu} \leqslant m_{\mu\xi}; m_{\xi\nu}$ by (7). Thus m satisfies 5.14(3).

Finally, we show that $\mu = \nu$ whenever $m_{\mu\nu} \leqslant 1'$, so that m satisfies 5.14(1). If μ, $\nu < \alpha$ then $m_{\mu\nu} = l_{\mu\nu} \leqslant 1'$, and $\mu = \nu$ since $l \in L_\alpha \mathfrak{A}$. Suppose $\mu < \alpha = \nu$ and $m_{\mu\nu} \leqslant 1'$. Then $x = m_{\kappa\alpha} \leqslant m_{\kappa\mu}; m_{\mu\alpha} = m_{\kappa\mu}; m_{\mu\nu} \leqslant m_{\kappa\mu}; 1' = m_{\kappa\mu}$, so $x = m_{\kappa\mu}$. Also $y = m_{\alpha\lambda} \leqslant m_{\alpha\mu}; m_{\mu\lambda} = m_{\nu\mu}; m_{\mu\lambda} = m_{\mu\nu}^{\cup}; m_{\mu\lambda} \leqslant 1'^{\cup}; m_{\mu\lambda} = m_{\mu\lambda}$, and thus $y = m_{\mu\lambda}$. But then $x = m_{\kappa\mu} = l_{\kappa\mu}$ and $y = m_{\mu\lambda} = l_{\mu\lambda}$, contradicting the hypothesis that $\langle \kappa, \lambda, x, y \rangle$ is a flaw in l.

LEMMA 5.17. *If $\mathfrak{A} \in WA$, $\mathfrak{A}$ satisfies 5.13(∗), and $l \in L_\alpha \mathfrak{A}$, then there are some $\beta \geqslant \alpha$ and some $m \in L_\beta \mathfrak{A}$ such that $l \subseteq m$ and if $\langle \kappa, \lambda, x, y \rangle$ is a flaw in m, then $\alpha \leqslant \kappa$, λ.*

PROOF. If l is complete, set $\beta = \alpha$ and $m = l$. Otherwise, let $\langle \langle \kappa_\xi, \lambda_\xi, x_\xi, y_\xi \rangle : \xi < \gamma \rangle$ be an enumeration of the flaws in l. Set $\beta = \alpha + \gamma$ and $m^0 = l$. Suppose $\delta < \gamma$ and m^δ has been constructed. If $\langle \kappa_\delta, \lambda_\delta, x_\delta, y_\delta \rangle$ is a flaw in m^δ, let $m^{\delta+1}$ be the $\mathfrak{A}$-labelling obtained by applying 5.16 to m^δ. For any limit ordinal $\mu < \gamma$, let $m^\mu = \bigcup_{\delta < \mu} m^\delta$. Finally, let $m = \bigcup_{\delta < \gamma} m^\delta$. It is then easy to show that $l \subseteq m \in L_\beta \mathfrak{A}$ and m has no flaws $\langle \kappa, \lambda, x, y \rangle$ with κ, $\lambda < \alpha$.

LEMMA 5.18. *If $\mathfrak{A} \in WA$, $\mathfrak{A}$ satisfies 5.13(∗), and $l \in L_\alpha \mathfrak{A}$, then there is some complete $m \in L_\beta \mathfrak{A}$ such that $l \subseteq m$.*

PROOF. Set $m^0 = l$, and for each $\kappa < \omega$ let $m^{\kappa+1}$ be the $\mathfrak{A}$-labelling obtained by applying 5.17 to the previously constructed labelling m^κ. Set $m = \bigcup_{\kappa < \omega} m^\kappa$. Then m is a complete $\mathfrak{A}$-labelling which contains l.

THEOREM 5.19. *If $\mathfrak{A} \in WA$, $\mathfrak{A}$ is atomic, and $\mathfrak{A}$ satisfies 5.13(∗), then $\mathfrak{A}$ is isomorphic to a subalgebra of $\mathfrak{Re} \, \alpha$, for some ordinal α.*

PROOF. Since $\mathfrak{A}$ satisfies 5.13(∗), there is a function a, mapping $I^{(\mathfrak{A})} \times I^{(\mathfrak{A})}$ into At $\mathfrak{A} \sim I^{(\mathfrak{A})}$, which satisfies 5.16(1)–(3). Let $\gamma = |I^{(\mathfrak{A})}|$, and let u be a one-to-one function mapping γ onto $I^{(\mathfrak{A})}$. Define $l \in {}^{\gamma \times \gamma}\text{At } \mathfrak{A}$ as follows, for all κ, $\lambda < \gamma$:

$$l_{\kappa\lambda} = \begin{cases} u_\kappa & \text{if } \kappa = \lambda, \\ a(u_\kappa, u_\lambda) & \text{if } \kappa < \lambda, \\ a(u_\kappa, u_\lambda)^{\cup} & \text{if } \lambda < \kappa. \end{cases}$$

Using 1.13(8), (16), 5.12(2), (3), and 5.16(1)–(3) one can easily prove that $l \in L_\gamma \mathfrak{A}$. By 5.18 there is some complete $m \in L_\alpha \mathfrak{A}$ such that $l \subseteq m$. To complete the proof it suffices, by 5.15, to show m is onto.

Let $x \in$ At $\mathfrak{A}$. If $x \in I^{(\mathfrak{A})}$ then $x = u_\kappa = l_{\kappa\kappa} = m_{\kappa\kappa}$ for some $\kappa < \gamma \leqslant \alpha$, so we may assume $x \notin I^{(\mathfrak{A})}$. Choose $\kappa, \lambda < \gamma$ so that $u_\kappa = x^d$ and $u_\lambda = x^r$. Then $a(x^d, x^r)$ $= l_{\kappa\lambda} = m_{\kappa\lambda}$. Set $y = a(x^r, x^r)$. Note that $y \in$ At $\mathfrak{A} \sim I^{(\mathfrak{A})}$, $m_{\kappa\lambda}^d = x^d$, $x^r = y^d$, and $m_{\kappa\lambda}^r = y^r$ by 5.16(1), (2). Consequently $m_{\kappa\lambda} \leqslant x ; y$ by 5.16(3). Since m is complete, there is some $\mu < \alpha$ such that $m_{\kappa\mu} = x$ and $m_{\mu\lambda} = y$. Thus x is in the range of m.

THEOREM 5.20. (1) *If $\mathfrak{A} \in WA$ and $\mathfrak{A}$ is complete and atomic, then $\mathfrak{A} \in$ Rl RRA.* (2) *$WA = $ SRl RRA.*

PROOF. (1) By 5.13, $\mathfrak{A} \cong \mathfrak{Rl}_a \mathfrak{B}$ for some complete atomic $\mathfrak{B} \in WA$ satisfying 5.13(∗), but $\mathfrak{B} \in RRA$ by 5.19 and 1.9(2), so $\mathfrak{A} \in I$ Rl $RRA = $ Rl I $RRA = $ Rl RRA by 5.7(2).

(2) Every WA is a subalgebra of a complete atomic WA by 4.2, so $WA \subseteq$ SRl RRA by part (1). On the other hand we have $RRA \subseteq WA$ by 1.9(1), 1.14, and 1.3, so SRl $RRA \subseteq$ SRl $WA = SWA = WA$ by 5.7(6), 5.8, and 1.5.

D. Resek proved an analogue of 5.20 for cylindric algebras in [15]. Henkin and Resek show in [5] that there are atomic subalgebras of relativized cylindric set algebras which are not relativized cylindric algebras. A study of their example and the connections between WA's, SA's, and cylindric algebras makes it seem possible that there are atomic WA's which are not relativized SA's.

Problem 5.21. Characterize Rl RRA. Is it an elementary class? Is every atomic WA in Rl SA? Is Rl $SA = $ Rl RRA?

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